

RELATIVE 2-SEGAL SPACES

MATTHEW B. YOUNG

ABSTRACT. We introduce a relative version of the 2-Segal simplicial spaces defined by Dyckerhoff and Kapranov and Gálvez-Carrillo, Kock and Tonks. Examples of relative 2-Segal spaces include the categorified unoriented cyclic nerve, real pseudo-holomorphic polygons in almost complex manifolds and the \mathcal{R}_\bullet -construction from Grothendieck-Witt theory. We show that a relative 2-Segal space defines a categorical representation of the Hall algebra associated to the base 2-Segal space. In this way, after decategorification we recover a number of known constructions of Hall algebra representations. We also describe some higher categorical interpretations of relative Segal spaces.

CONTENTS

Introduction	1
1. Higher Segal spaces	4
1.1. Simplicial objects	4
1.2. 1-Segal spaces	4
1.3. 2-Segal spaces	5
1.4. The Waldhausen \mathcal{S}_\bullet -construction	6
2. Relative higher Segal spaces	7
2.1. Relative 1-Segal spaces	7
2.2. Relative 2-Segal spaces	10
2.3. Symmetric polyhedral subdivisions	13
2.4. Stable framed objects	15
2.5. Real pseudo-holomorphic polygons	17
3. Relative Segal spaces from categories with dualities	20
3.1. Categories with duality	20
3.2. Unoriented categorified nerves	21
3.3. Unoriented categorified twisted cyclic nerves	22
3.4. The \mathcal{R}_\bullet -construction	25
4. Applications	28
4.1. Categorical Hall algebra representations	28
4.2. Hall monoidal module categories	33
4.3. Modules over multivalued categories	35
References	39

INTRODUCTION

Motivated by Segal's notion of a Γ -space [35], Rezk introduced Segal spaces in his study of the homotopy theory of $(\infty, 1)$ -categories [29]. Generalizing these ideas, for each integer $k \geq 1$, Dyckerhoff and Kapranov introduced k -Segal spaces [5]. Very

Date: November 29, 2016.

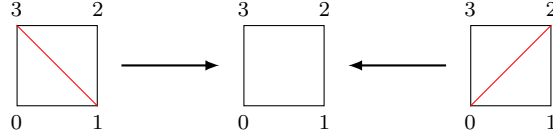
2010 Mathematics Subject Classification. Primary: 18G30; Secondary 19G38, 16G20.

Key words and phrases. Higher Segal spaces. Categorified Hall algebra representations. Categories with duality. Grothendieck-Witt theory.

roughly, a simplicial topological space X_\bullet is called k -Segal if it satisfies a collection of locality conditions governed by polyhedral subdivisions of a k -dimensional cyclic polytope. When $k = 1$, so that the locality conditions are governed by subdivisions of the interval, the 1-Segal conditions state that, for each $n \geq 2$, the canonical map to the homotopy fibre product

$$X_n \rightarrow \overbrace{X_1 \times_{X_0}^R \cdots \times_{X_0}^R X_1}^{n \text{ factors}}$$

is a weak homotopy equivalence, thus recovering Rezk's Segal spaces. The 2-Segal spaces, which were introduced independently by Gálvez-Carrillo, Kock and Tonks [10] under the name decomposition spaces, obey locality conditions governed by polyhedral subdivisions of convex plane polygons. The first non-trivial conditions on a 2-Segal space derive from the two triangulations of the square,



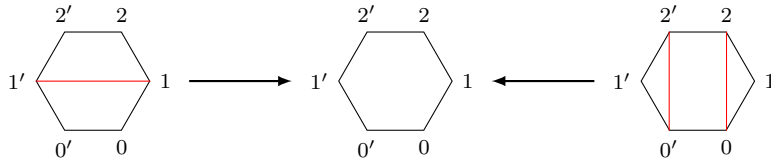
and state that the induced morphisms

$$X_{\{0,1,3\}} \times_{X_{\{1,3\}}}^R X_{\{1,2,3\}} \leftarrow X_{\{0,1,2,3\}} \rightarrow X_{\{0,1,2\}} \times_{X_{\{0,2\}}}^R X_{\{0,2,3\}} \quad (1)$$

are weak equivalences. A large number of examples of 2-Segal spaces from a diverse range of subjects are described in [5] and [10].

One motivation to study 2-Segal spaces is the theory of Hall algebras. Indeed, as exploited by Dyckerhoff and Kapranov [5], the 2-Segal conditions admit a natural interpretation as higher coherence conditions on a multiplication defined on the 1-simplices X_1 , the weak equivalences (1) imposing weak associativity. From the Hall algebra point of view, the most important example of a 2-Segal space is the Waldhausen \mathcal{S}_\bullet -construction of an exact category [43]. Applying suitable realization functors to this 2-Segal space recovers various familiar incarnations of the Hall algebra [30], [23], [17], [21]. However, these realizations use only the lowest 2-Segal conditions, namely (1). Taking into account the remaining conditions leads to higher categorical structures and thus to categorical Hall algebras. Other applications of 2-Segal spaces are described in [6], [7], [10].

Partially motivated by the representation theory of Hall algebras, in this paper we introduce relative higher Segal spaces. For an integer $k \geq 1$, a relative k -Segal space over a k -Segal space X_\bullet is a $(k-1)$ -Segal space Y_\bullet together with a morphism $Y_\bullet \rightarrow X_\bullet$ which satisfies k -dimensional locality conditions involving both X_\bullet and Y_\bullet ; by convention the 0-Segal conditions are vacuous. In the simplest case, the right relative 1-Segal conditions are equivalent to the condition that $Y_\bullet \rightarrow X_\bullet$ be a right fibration of 1-Segal spaces in the sense of [19], [3]. The relative 2-Segal conditions are cleanly formulated in terms of reflection symmetric polyhedral subdivisions of symmetric plane polygons, the most basic of which are the two subdivisions of the plane hexagon,



and translate into the requirement that the canonical morphisms

$$Y_{\{0,1\}} \times_{Y_{\{1\}}}^R Y_{\{1,2\}} \leftarrow Y_2 \rightarrow X_{\{0,1,2\}} \times_{X_{\{0,2\}}}^R Y_{\{0,2\}} \quad (2)$$

be weak equivalences. Similar to the case of 2-Segal spaces, the relative 2-Segal conditions give higher coherence conditions for appropriately defined left and right actions of X_1 on the 0-simplices Y_0 . From this point of view, the weak equivalences (2) are the weak module associativity constraints. Relative 2-Segal spaces therefore lead naturally to categorical representations of the Hall algebra associated to X_\bullet ; see Theorems 4.2 and 4.3 for particular instances of this construction. In this way we obtain natural categorifications of many of the Hall algebra representations which have appeared in the literature. For example, we prove that a stable framed variant of the Waldhausen \mathcal{S}_\bullet -construction is relative 2-Segal space over the ordinary \mathcal{S}_\bullet -construction, thus categorifying the Hall algebra representations studied in [38], [9]; see Theorem 2.7. We also prove that the \mathcal{R}_\bullet -construction from Grothendieck-Witt theory (i.e. algebraic KR -theory) [36], [12] is relative 2-Segal over the \mathcal{S}_\bullet -construction; see Theorem 3.6. The input for the \mathcal{R}_\bullet -construction is a proto-exact category with duality which satisfies a reduction assumption. In the case of exact categories the \mathcal{R}_\bullet -construction categorifies the Hall algebra representations of [8], [44], [45] while for the proto-exact category $\text{Rep}_{\mathbb{F}_1}(Q)$ of representations of a quiver over \mathbb{F}_1 we obtain new modules over Szczesny's Hall algebras [39]. The latter modules will be the subject of future work.

We also give examples of relative 2-Segal spaces which do not come from known Hall algebra representations. Starting from an almost complex manifold M with a real structure, we construct in Theorem 2.8 a relative 2-Segal semi-simplicial set consisting of real pseudo-holomorphic polygons in M , the base 2-Segal set being the pseudo-holomorphic polygons in M [5]. In Theorem 3.5 we prove that the categorified unoriented twisted cyclic nerve of a category with endomorphism and compatible duality structure is relative 2-Segal over the categorified twisted cyclic nerve. This example can be viewed as a homotopical incarnation of the unoriented loop space of an orbifold.

A common theme of many of the relative 2-Segal spaces in this paper is that they are in a sense unoriented. It is tempting to view these examples in the context of orientifold string theory. (The stable framed \mathcal{S}_\bullet -construction is different, being related to string theory with defects.) A general feature of the orientifold construction is that it imposes \mathbb{Z}_2 -equivariance conditions on objects in the parent string theory, such as reduction of structure groups of vector bundles from general linear to orthogonal or symplectic groups, as in the \mathcal{R}_\bullet -construction, or replacing oriented string worldsheets with unoriented worldsheets, similar to the real pseudo-holomorphic polygon and unoriented nerve constructions. In [5, Remark 3.7.8] it is speculated that there exists a sort of mirror symmetry relating the 2-Segal spaces arising from the \mathcal{S}_\bullet -construction with those arising from the pseudo-holomorphic polygon construction. It is natural to speculate that such a mirror symmetry admits an orientifold enhancement, relating the \mathcal{R}_\bullet - and real pseudo-holomorphic polygon constructions.

Higher categorical interpretations of (higher) Segal spaces have been given in [29], [16], [5], two of which we lift to the relative setting. It is a classical fact that 1-Segal simplicial sets can be characterized as the essential image of the fully faithful nerve functor $N_\bullet : \text{Cat} \rightarrow \text{Set}_\Delta$. In a similar vein, right relative 1-Segal simplicial sets are the essential image of the relative nerve construction applied to the category of discrete right fibrations, which via the Grothendieck construction we interpret as presheaves on small categories; see Proposition 2.2. Using the work of several authors [14], [22], [3] we explain a quasicategorical generalization of these statements by considering instead right relative 1-Segal combinatorial simplicial

spaces. Secondly, in [5] it is proved that 2-Segal simplicial sets are equivalent to the category of multivalued categories. Pursuing an interpretation in terms of actions of categories as in the 1-Segal case, we prove that the category of relative 2-Segal simplicial sets is equivalent to the category of modules over multivalued categories; see Theorem 4.5.

Remark. After this paper was completed, a pre-print by Tashi Walde [42] was posted to the arXiv which also aims at developing a theory of modules over higher Segal spaces.

Acknowledgements. The author would like to thank Ajneet Dhillon, Tobias Dyckerhoff, Karol Szumilo and Pal Zsamboki for interesting discussions regarding the subject of this paper.

1. HIGHER SEGAL SPACES

In this section we recall, following closely [5], some required background material from the theory of higher Segal spaces.

1.1. Simplicial objects. Let Δ be the category whose objects are the non-empty finite ordinals $[n] = \{0, \dots, n\}$, $n \geq 0$, and whose morphisms are weakly monotone set maps. We sometimes consider the object $[n] \in \Delta$ as a category itself. Explicitly, the objects of $[n]$ are labelled by integers $0 \leq k \leq n$ and the morphism set $\text{Hom}_{[n]}(i, j)$ is empty if $i > j$ and consists of a single element if $i \leq j$. Let also $\Delta_{\text{inj}} \subset \Delta$ be the subcategory of injective morphisms.

Let \mathcal{C} be a category. A simplicial object of \mathcal{C} is a functor $X_{\bullet} : \Delta^{\text{op}} \rightarrow \mathcal{C}$. We write X_n for $X_{[n]} \in \mathcal{C}$ if it will not lead to confusion. The face and degeneracy maps of X_{\bullet} are denoted by

$$\partial_i : X_n \rightarrow X_{n-1}, \quad s_i : X_n \rightarrow X_{n+1}, \quad 0 \leq i \leq n$$

respectively. More generally, a semi-simplicial object of \mathcal{C} is a functor $X_{\bullet} : \Delta_{\text{inj}}^{\text{op}} \rightarrow \mathcal{C}$. Since any non-empty totally ordered finite set I is canonically isomorphic to $[n]$ for some $n \geq 0$, given a semi-simplicial object X_{\bullet} we can unambiguously define X_I . We write Δ^I for the simplicial set $\text{Hom}_{\Delta}(-, I)$. In particular, we set $\Delta^n = \Delta^{[n]}$.

Given categories \mathcal{C} and \mathcal{D} , denote by $[\mathcal{C}, \mathcal{D}]$ the category of functors $\mathcal{C} \rightarrow \mathcal{D}$. Let **Set**, **Grpd** and **Top** be the categories of sets, small groupoids and compactly generated topological spaces. Objects of the categories $[\Delta^{\text{op}}, \text{Top}]$ and $[\Delta^{\text{op}}, \text{Grpd}]$ are called simplicial spaces and groupoids while objects of $\mathbb{S} = [\Delta^{\text{op}}, \text{Set}]$ and $\mathbb{S}_{\Delta} = [\Delta^{\text{op}}, \mathbb{S}]$ are called simplicial sets and combinatorial simplicial spaces, respectively.

1.2. 1-Segal spaces. Segal spaces (called 1-Segal spaces below) were introduced by Rezk [29, §4] (see also [35, §1]). The definition below is slightly different, omitting a fibrancy condition. Write $- \times_U^R -$ for the homotopy fibre product over a topological space U .

Definition. A semi-simplicial space $X_{\bullet} : \Delta_{\text{inj}}^{\text{op}} \rightarrow \text{Top}$ is called 1-Segal if for every $n \geq 2$ the map

$$X_n \rightarrow X_1 \times_{X_0}^R \cdots \times_{X_0}^R X_1$$

induced by the inclusions $\{i, i+1\} \hookrightarrow [n]$, $i = 0, \dots, n-1$, is a weak homotopy equivalence.

It is straightforward to verify that a semi-simplicial space X_{\bullet} is 1-Segal if and only if one of the following two conditions hold:

- (i) For every $n \geq 2$ and every $0 \leq i_1 < \dots < i_l \leq n$ the map

$$X_n \rightarrow X_{i_1} \times_{X_{\{i_1\}}}^R \dots \times_{X_{\{i_l\}}}^R X_{n-i_l}$$

induced by the inclusions $\{0, \dots, i_1\}, \dots, \{i_l, \dots, n\} \hookrightarrow [n]$ is a weak equivalence.

- (ii) For every $n \geq 2$ and every $0 \leq i \leq n$ the map

$$X_n \rightarrow X_{\{0, \dots, i\}} \times_{X_{\{i\}}}^R X_{\{i, \dots, n\}}$$

induced by the inclusions $\{0, \dots, i\}, \{i, \dots, n\} \hookrightarrow [n]$ is a weak equivalence.

With only minor changes one can formulate the theory of 1-Segal objects (along with the higher and relative variants defined below) of a combinatorial model category \mathcal{C} [5, §5], in which case $-\times_{-}^R-$ is a homotopy limit. So, for example, we can speak of 1-Segal simplicial sets, groupoids or combinatorial simplicial spaces.

Example. The nerve $N_{\bullet}(\mathcal{C})$ of a small category \mathcal{C} is the simplicial set which assigns to $[n] \in \Delta$ the set underlying the functor category $[[n], \mathcal{C}]$. It is well-known that $N_{\bullet}(\mathcal{C})$ is a 1-Segal simplicial set or, alternatively, a discrete 1-Segal simplicial space. In fact, the nerve functor $N_{\bullet} : \mathbf{Cat} \rightarrow \mathbb{S}$ is fully faithful with essential image the 1-Segal simplicial sets. The category \mathcal{X} associated to a 1-Segal simplicial set X_{\bullet} has X_0 as its set of objects and

$$\mathrm{Hom}_{\mathcal{X}}(x_0, x_1) = \{x_0\} \times_{X_{\{0\}}} X_{\{0,1\}} \times_{X_{\{1\}}} \{x_1\}$$

as its morphism sets. Composition of morphisms is defined using the lowest 1-Segal conditions while associativity follows from the higher 1-Segal conditions. \triangleleft

Example. The categorified nerve $\mathcal{N}_{\bullet}(\mathcal{C})$ of a small category \mathcal{C} is the 1-Segal simplicial groupoid which assigns to $[n] \in \Delta$ the maximal groupoid of the category $[[n], \mathcal{C}]$ [29, §3.5]. Passing to classifying spaces gives a 1-Segal simplicial space $B\mathcal{N}_{\bullet}(\mathcal{C})$. In Rezk's framework the categorified nerve is preferred to the ordinary nerve as the former gives a complete 1-Segal space. \triangleleft

1.3. 2-Segal spaces. The 1-Segal spaces are the first in an infinite tower of higher Segal spaces introduced by Dyckerhoff and Kapranov [5]; see also [10]. In this section we focus on the next step in this tower.

For each $n \geq 2$ let $P_n \subset \mathbb{R}^2$ be a convex $(n+1)$ -gon with a total order on its vertices that is consistent with the counterclockwise orientation of \mathbb{R}^2 . The total order induces a canonical identification of the set of vertices of P_n with $[n]$. Let \mathcal{P} be a polyhedral subdivision of P_n . Associating to each polygon of \mathcal{P} its set of vertices defines a collection of subsets $[n]$. Write $\Delta^{\mathcal{P}} \subset \Delta^n$ for the corresponding simplicial subset.

Let X_{\bullet} be a semi-simplicial space. A polyhedral subdivision \mathcal{P} of P_n induces a map

$$f_{\mathcal{P}} : X_n \simeq (\Delta^n, X_{\bullet})_R \rightarrow (\Delta^{\mathcal{P}}, X_{\bullet})_R$$

where, following [5, §2.2], for a semi-simplicial set D the membrane space is

$$(D, X_{\bullet})_R = \mathrm{holim}_{\leftarrow \{\Delta^{\mathcal{P}} \hookrightarrow D\} \in \Delta_{\mathrm{inj}}/D}^{\mathrm{Top}} X_{\mathcal{P}}.$$

Definition. A semi-simplicial space X_{\bullet} is called 2-Segal if for every $n \geq 3$ and every triangulation \mathcal{T} of P_n the map $f_{\mathcal{T}} : X_n \rightarrow (\Delta^{\mathcal{T}}, X_{\bullet})_R$ is a weak equivalence.

As in the case of 1-Segal spaces, the 2-Segal conditions can be checked using coarser subdivisions of P_n . Indeed, it is proved in [5, Proposition 2.3.2] that X_{\bullet} is 2-Segal if and only if one of the following conditions hold:

- (i) For every $n \geq 3$ and every polyhedral subdivision \mathcal{P} of P_n the map $f_{\mathcal{P}} : X_n \rightarrow (\Delta^{\mathcal{P}}, X_{\bullet})_R$ is a weak equivalence.

(ii) For every $n \geq 3$ and every $0 \leq i < j \leq n$ the map

$$X_n \rightarrow X_{\{i, \dots, j\}} \times_{X_{\{i, j\}}}^R X_{\{0, \dots, i, j, \dots, n\}} \quad (3)$$

induced by the inclusions $\{i, \dots, j\}, \{0, \dots, i, j, \dots, n\} \hookrightarrow [n]$ is a weak equivalence.

(iii) The maps (3) are weak equivalences if $i = 0$ or $j = n$.

The following definition uses degeneracy maps and so can only be formulated in the simplicial setting.

Definition. A 2-Segal simplicial space X_\bullet is called *unital 2-Segal* if for every $n \geq 2$ and every $0 \leq i \leq n-1$ the diagram

$$\begin{array}{ccc} X_{n-1} & \xrightarrow{s_i} & X_n \\ \downarrow & & \downarrow \\ X_{\{i\}} & \longrightarrow & X_{\{i, i+1\}} \end{array} \quad (4)$$

is homotopy Cartesian.

One simple construction of 2-Segal spaces is the following.

Proposition 1.1 ([5, Propositions 2.3.3, 2.5.3], [10, Proposition 3.5]). *Let X_\bullet be a 1-Segal semi-simplicial space. Then X_\bullet is 2-Segal. If in fact X_\bullet is a simplicial space, then X_\bullet is unital 2-Segal.*

1.4. The Waldhausen \mathcal{S}_\bullet -construction. We recall a motivating example of a unital 2-Segal space. We work with proto-exact categories, a not-necessarily additive generalization of exact categories in the sense of Quillen [28].

Definition ([5, §2.4]). *A proto-exact category is a pointed category \mathcal{C} , with zero object 0, together with two classes of morphisms, \mathfrak{I} and \mathfrak{D} , called inflations and deflations and denoted by \rightarrowtail and \twoheadrightarrow , respectively, having the following properties:*

- (i) *Any morphism $0 \rightarrowtail U$ is in \mathfrak{I} and any morphism $U \twoheadrightarrow 0$ is in \mathfrak{D} .*
- (ii) *The classes \mathfrak{I} and \mathfrak{D} are closed under composition and contain all isomorphisms.*
- (iii) *A commutative square of the form*

$$\begin{array}{ccc} U & \rightarrowtail & V \\ \downarrow & & \downarrow \\ W & \rightarrowtail & X \end{array} \quad (5)$$

is Cartesian if and only if it is coCartesian.

- (iv) *Any diagram $W \rightarrowtail X \leftarrow V$ can be completed to a biCartesian diagram of the form (5).*
- (v) *Any diagram $W \leftarrow U \rightarrowtail V$ can be completed to a biCartesian diagram of the form (5).*

BiCartesian squares of the form

$$\begin{array}{ccc} U & \rightarrowtail & V \\ \downarrow & & \downarrow \\ 0 & \rightarrowtail & X \end{array}$$

are called conflations and play the role of short exact sequences in \mathcal{C} . Familiar examples of proto-exact categories include abelian and exact categories. A more exotic example is given by the category of representations of a quiver over \mathbb{F}_1 , as described in [39].

The Waldhausen \mathcal{S}_\bullet -construction associates to a proto-exact category \mathcal{C} a simplicial groupoid $\mathcal{S}_\bullet(\mathcal{C})$ as follows [43, §1.3], [5, §2.4]. Let $\mathbf{Ar}_n = [[1], [n]]$ be the arrow category of $[n]$. The assignment $[n] \mapsto \mathbf{Ar}_n$ defines a cosimplicial category. An object $\{(i \rightarrow j) \mapsto A_{\{i,j\}}\}_{0 \leq i \leq j \leq n}$ of the functor category $[\mathbf{Ar}_n, \mathcal{C}]$ is a commutative diagram in \mathcal{C} of the form

$$\begin{array}{ccccccc}
 A_{\{0,0\}} & \longrightarrow & A_{\{0,1\}} & \longrightarrow & \cdots & \longrightarrow & A_{\{0,n-1\}} & \longrightarrow & A_{\{0,n\}} \\
 & & \downarrow & & & & \downarrow & & \downarrow \\
 & & A_{\{1,1\}} & \longrightarrow & \cdots & \longrightarrow & A_{\{1,n-1\}} & \longrightarrow & A_{\{1,n\}} \\
 & & & & \ddots & & \vdots & & \vdots \\
 & & & & & & \downarrow & & \downarrow \\
 & & & & & & A_{\{n-1,n-1\}} & \longrightarrow & A_{\{n-1,n\}} \\
 & & & & & & & & \downarrow \\
 & & & & & & & & A_{\{n,n\}}
 \end{array}$$

Let $\mathcal{W}_n(\mathcal{C}) \subset [\mathbf{Ar}_n, \mathcal{C}]$ be the full subcategory consisting of diagrams which have the following properties:

- (i) For every $0 \leq i \leq n$ the object $A_{\{i,i\}}$ is isomorphic to $0 \in \mathcal{C}$.
- (ii) All horizontal morphisms are inflations and all vertical morphisms are deflations.
- (iii) Each square that can be formed in the diagram is biCartesian.

Let $\mathcal{S}_n(\mathcal{C})$ be the maximal groupoid of $\mathcal{W}_n(\mathcal{C})$. Then $\mathcal{S}_\bullet(\mathcal{C})$ is a simplicial groupoid, the degeneracy map $s_i : \mathcal{S}_n(\mathcal{C}) \rightarrow \mathcal{S}_{n+1}(\mathcal{C})$ inserting a row/column of identity morphisms after the i th row/column and the face map $\partial_i : \mathcal{S}_n(\mathcal{C}) \rightarrow \mathcal{S}_{n-1}(\mathcal{C})$ deleting the i th row/column and composing the obvious morphisms.

Theorem 1.2 ([5, Proposition 2.4.8], [10, Theorem 10.14]). *For any proto-exact category \mathcal{C} , the simplicial space $B\mathcal{S}_\bullet(\mathcal{C})$ is unital 2-Segal.*

When \mathcal{C} is an exact category the simplicial space $B\mathcal{S}_\bullet(\mathcal{C})$ plays a fundamental role in the higher algebraic K -theory of \mathcal{C} . Indeed, we have $K_i(\mathcal{C}) = \pi_i \Omega |B\mathcal{S}_\bullet(\mathcal{C})|$ where we take 0 as the basepoint of $|B\mathcal{S}_\bullet(\mathcal{C})|$. See [40], [43].

A variation of the Waldhausen \mathcal{S}_\bullet -construction was defined in [1], giving a functor from the category of augmented stable double categories to the category of simplicial sets. Moreover, it was proved that this functor is fully faithful with essential image the unital 2-Segal simplicial sets.

2. RELATIVE HIGHER SEGAL SPACES

2.1. Relative 1-Segal spaces. As motivation for relative 2-Segal spaces, in this section we study relative 1-Segal spaces. Throughout this section we fix a 1-Segal semi-simplicial space X_\bullet .

Definition. A morphism $F_\bullet : Y_\bullet \rightarrow X_\bullet$ of semi-simplicial spaces is called *right relative 1-Segal* if for every $n \geq 1$ and every $0 \leq i \leq n$ the outside square of the

diagram

$$\begin{array}{ccc}
 Y_n & \longrightarrow & Y_{\{i, \dots, n\}} \\
 \downarrow & & \downarrow \\
 Y_{\{0, \dots, i\}} & \longrightarrow & Y_{\{i\}} \\
 F_{\{0, \dots, i\}} \downarrow & & \downarrow F_{\{i\}} \\
 X_{\{0, \dots, i\}} & \longrightarrow & X_{\{i\}}
 \end{array} \tag{6}$$

is homotopy Cartesian.

Similarly, left relative 1-Segal spaces are morphisms $F_\bullet : Y_\bullet \rightarrow X_\bullet$ of semi-simplicial spaces for which the outside square of the diagram

$$\begin{array}{ccccc}
 Y_n & \longrightarrow & Y_{\{i, \dots, n\}} & \xrightarrow{F_{\{i, \dots, n\}}} & X_{\{i, \dots, n\}} \\
 \downarrow & & \downarrow & & \downarrow \\
 Y_{\{0, \dots, i\}} & \longrightarrow & Y_{\{i\}} & \xrightarrow{F_{\{i\}}} & X_{\{i\}}
 \end{array}$$

is homotopy Cartesian. All results below will be formulated for right relative 1-Segal spaces; analogous results hold for left relative 1-Segal spaces.

Example. Let X_\bullet be a 1-Segal semi-simplicial space. Then the identity morphism $1_{X_\bullet} : X_\bullet \rightarrow X_\bullet$ is both left and right relative 1-Segal. \triangleleft

Example. Let x_∞ be an object of a small category \mathcal{C} and let $\mathcal{C}_{/x_\infty}$ be the corresponding overcategory. The forgetful functor $\mathcal{C}_{/x_\infty} \rightarrow \mathcal{C}$ induces a simplicial morphism $N_\bullet(\mathcal{C}_{/x_\infty}) \rightarrow N_\bullet(\mathcal{C})$ which is right relative 1-Segal. Using instead the undercategory $_{x_\infty}\mathcal{C}$ gives a left relative 1-Segal simplicial set over $N_\bullet(\mathcal{C})$. \triangleleft

Example. Suppose that a group G acts on a set E . Then G acts diagonally on the Cartesian product E^n , $n \geq 1$. The action groupoid $G \parallel E^{n+1}$ has object set E^{n+1} while the morphism set $\text{Hom}_{G \parallel E^{n+1}}(e_\bullet, e'_\bullet)$ consists of those $g \in G$ for which $g \cdot e_\bullet = e'_\bullet$. The assignment $[n] \mapsto G \parallel E^{n+1}$ defines a simplicial groupoid $\mathcal{S}_\bullet(G, E)$, the face (resp. degeneracy) maps omitting (resp. repeating) the appropriate entries of $E^{\bullet+1}$. Passing to classifying spaces yields a 1-Segal space $B\mathcal{S}_\bullet(G, E)$, called the Hecke-Waldhausen space [5, §2.6].

Suppose now that $H \leq G$ is a subgroup. The inclusion $H \hookrightarrow G$ defines a simplicial morphism $\mathcal{S}_\bullet(H, E) \rightarrow \mathcal{S}_\bullet(G, E)$. At the level of classifying spaces we obtain a morphism $B\mathcal{S}_\bullet(H, E) \rightarrow B\mathcal{S}_\bullet(G, E)$ which is both left and right relative 1-Segal; this can be verified in much the same way as the 1-Segal property of $B\mathcal{S}_\bullet(G, E)$. At the level of geometric realizations, the latter map is homotopy equivalent to the canonical morphism $BH \rightarrow BG$ of classifying spaces (cf. [5, Proposition 2.6.7]). \triangleleft

We require the following alternate characterization of right relative 1-Segal spaces.

Proposition 2.1. *A semi-simplicial morphism $F_\bullet : Y_\bullet \rightarrow X_\bullet$ is right relative 1-Segal if and only if Y_\bullet is 1-Segal and the map*

$$(F_1, \partial_0) : Y_1 \rightarrow X_1 \times_{X_0}^R Y_0 \tag{7}$$

is a weak equivalence.

Proof. Suppose that F_\bullet is right relative 1-Segal. Taking $i = n = 1$ in diagram (6) implies that the map (7) is a weak equivalence. For general i and n , both the outside square and the bottom square (which is a degenerate version of the outside

square) of diagram (6) are homotopy Cartesian. By the 2-out-of-3 property of weak equivalences the top square is then also homotopy Cartesian. Hence Y_\bullet is 1-Segal.

Conversely, suppose that Y_\bullet is 1-Segal and that the map (7) is a weak equivalence. We then have the following sequence of weak equivalences:

$$\begin{aligned}
Y_n &\xrightarrow{\text{w.e.}} Y_{\{0,\dots,i\}} \times_{Y_{\{i\}}}^R Y_{\{i,\dots,n\}} \\
&\xrightarrow{\text{w.e.}} Y_{\{0,1\}} \times_{Y_{\{1\}}}^R Y_{\{1,\dots,i\}} \times_{Y_{\{i\}}}^R Y_{\{i,\dots,n\}} \\
&\xrightarrow{\text{w.e.}} X_{\{0,1\}} \times_{X_{\{1\}}}^R Y_{\{1\}} \times_{Y_{\{1\}}}^R Y_{\{1,\dots,i\}} \times_{Y_{\{i\}}}^R Y_{\{i,\dots,n\}} \\
&\vdots \\
&\xrightarrow{\text{w.e.}} X_{\{0,1\}} \times_{X_{\{1\}}}^R \cdots \times_{X_{\{i-1\}}}^R X_{\{i-1,i\}} \times_{X_{\{i\}}}^R Y_{\{i,\dots,n\}} \\
&\xleftarrow{\text{w.e.}} X_{\{0,\dots,i\}} \times_{X_{\{i\}}}^R Y_{\{i,\dots,n\}}.
\end{aligned}$$

The right relative 1-Segal map $Y_n \rightarrow X_{\{0,\dots,i\}} \times_{X_{\{i\}}}^R Y_{\{i,\dots,n\}}$ makes this chain of maps commute and is thus a weak equivalence by iterated application of the 2-out-of-3 property of weak equivalences. Hence F_\bullet is right relative 1-Segal. \square

Kazhdan and Varshavsky [19] and de Brito [3] define a right 1-Segal fibration to be a morphism of 1-Segal spaces $F_\bullet : Y_\bullet \rightarrow X_\bullet$ for which the map (7) is a weak equivalence. Regarding nomenclature, it is proved in [3, Proposition 1.10] that right 1-Segal fibrations in \mathbb{S}_Δ are fibrant objects of a natural left Bousfield localization of $(\mathbf{Seg}_1)_{/X_\bullet}$, the model structure on the overcategory $(\mathbb{S}_\Delta)_{/X_\bullet}$ induced by Rezk's 1-Segal model structure \mathbf{Seg}_1 on \mathbb{S}_Δ [29, Theorem 7.1]. Proposition 2.1 implies that right relative 1-Segal spaces and right 1-Segal fibrations are equivalent notions.

Right relative 1-Segal simplicial sets admit a simple nerve theoretic characterization, generalizing that of 1-Segal simplicial sets. To formulate this, given a small category \mathcal{X} write $\mathbf{DRFib}(\mathcal{X})$ for the full subcategory of $\mathbf{Cat}_{/\mathcal{X}}$ consisting of discrete right fibrations. Similarly for a 1-Segal simplicial set X_\bullet let $\mathbf{1-SegRelS}_{/X_\bullet}$ be the full subcategory of $\mathbb{S}_{/X_\bullet}$ consisting of right relative 1-Segal simplicial sets.

Proposition 2.2. *Let \mathcal{X} be a small category. The relative nerve functor*

$$N_\bullet^{\text{rel}} : \mathbf{Cat}_{/\mathcal{X}} \rightarrow \mathbb{S}_{/N_\bullet(\mathcal{X})}, \quad (\mathcal{Y} \xrightarrow{F} \mathcal{X}) \mapsto (N_\bullet(\mathcal{Y}) \xrightarrow{N_\bullet(F)} N_\bullet(\mathcal{X}))$$

is fully faithful and fits into the commutative diagram of functors

$$\begin{array}{ccccc}
[\mathcal{X}^{\text{op}}, \mathbf{Set}] & \xrightarrow{f} & \mathbf{Cat}_{/\mathcal{X}} & \xrightarrow{N_\bullet^{\text{rel}}} & \mathbb{S}_{/N_\bullet(\mathcal{X})} \\
& \searrow \sim & \uparrow & & \uparrow \\
& & \mathbf{DRFib}(\mathcal{X}) & \xrightarrow{\sim} & \mathbf{1-SegRelS}_{/N_\bullet(\mathcal{X})}
\end{array}$$

with indicated equivalences, where f denotes the Grothendieck construction. In particular, there is an equivalence of categories $\mathbf{1-SegRelS}_{/N_\bullet(\mathcal{X})} \simeq [\mathcal{X}^{\text{op}}, \mathbf{Set}]$.

Proof. Commutativity of the triangle in the above diagram is standard. That N_\bullet^{rel} is fully faithful is well-known; see for example [34, Proposition 2.1]. To see that N_\bullet^{rel} restricts to the claimed equivalence, let $(F : \mathcal{Y} \rightarrow \mathcal{X}) \in \mathbf{DRFib}(\mathcal{X})$. Then $N_\bullet(F) : N_\bullet(\mathcal{Y}) \rightarrow N_\bullet(\mathcal{X})$ is a morphism of 1-Segal simplicial sets. The condition that F is a discrete right fibration is precisely the condition that the map $N_1(\mathcal{Y}) \rightarrow N_1(\mathcal{X}) \times_{N_{\{1\}}(\mathcal{X})} N_{\{1\}}(\mathcal{Y})$ is a bijection. The other direction is similar. \square

In other words, right relative 1-Segal spaces over $N_\bullet(\mathcal{X}) \in \mathbb{S}$ are equivalent to presheaves on \mathcal{X} . Pursuing this perspective, suppose now that $X_\bullet \in \mathbb{S}_\Delta$ is a complete 1-Segal combinatorial simplicial space and let $F_\bullet : Y_\bullet \rightarrow X_\bullet$ be right

relative 1-Segal. The quasicategory \mathcal{X} modelled by X_\bullet (see [29, §5], [16, §4]) has object set the 0-simplices of X_0 and has mapping spaces

$$\mathrm{map}_{\mathcal{X}}(x_0, x_1) = \{x_0\} \times_{X_{\{0\}}}^R X_{\{0,1\}} \times_{X_{\{1\}}}^R \{x_1\} \in \mathbb{S}.$$

Here $\{x\}$ is regarded as the simplicial set Δ^0 . The lowest 1-Segal conditions define, up to homotopy, a composition law

$$\mathrm{map}_{\mathcal{X}}(x_0, x_1) \times \mathrm{map}_{\mathcal{X}}(x_1, x_2) \rightarrow \mathrm{map}_{\mathcal{X}}(x_0, x_2)$$

which by the remaining 1-Segal conditions is coherently associative. In particular, the homotopy category $\mathrm{ho}(\mathcal{X})$ is a genuine category. Continuing, for each $x \in \mathcal{X}$ define a Kan complex by $\mathcal{F}(x) = \{x\} \times_{X_0}^R Y_0 \in \mathbb{S}$. The diagram

$$\begin{array}{ccc} \{x_0\} \times_{X_{\{0\}}}^R Y_{\{0,1\}} & \xrightarrow{\quad\quad\quad} & \{x_0\} \times_{X_{\{0\}}}^R Y_{\{0\}} \\ \downarrow \text{w.e.} & & \\ \{x_0\} \times_{X_{\{0\}}}^R X_{\{0,1\}} \times_{X_{\{1\}}}^R Y_{\{1\}} & & \\ \uparrow & & \\ (\{x_0\} \times_{X_{\{0\}}}^R X_{\{0,1\}} \times_{X_{\{1\}}}^R \{x_1\}) \times (\{x_1\} \times_{X_{\{1\}}}^R Y_{\{1\}}) & & \end{array}$$

whose indicated arrow is a weak equivalence by the lowest right relative 1-Segal condition, defines up to homotopy an action map

$$\mathrm{map}_{\mathcal{X}}(x_0, x_1) \times \mathcal{F}(x_1) \rightarrow \mathcal{F}(x_0).$$

The remaining relative 1-Segal conditions ensure that this action is coherently associative. By combining [22, Proposition 5.1.1.1] (see also [14]) and [3, Theorem 1.22] we see that, up to weak equivalence, any $(\infty, 1)$ -presheaf on \mathcal{X} arises in this way.

2.2. Relative 2-Segal spaces. In this section we give a direct definition of relative 2-Segal spaces. In Section 2.3 we describe a second approach using polyhedral subdivisions.

Throughout this section we fix a 2-Segal semi-simplicial space X_\bullet .

Definition. A morphism $F_\bullet : Y_\bullet \rightarrow X_\bullet$ of semi-simplicial spaces is called *relative 2-Segal* if

- (1) for every $n \geq 2$ and every $0 \leq i < j \leq n$ the outside square of the diagram

$$\begin{array}{ccc} Y_n & \xrightarrow{\quad\quad\quad} & Y_{\{0,\dots,i,j,\dots,n\}} \\ \downarrow & & \downarrow \\ Y_{\{i,\dots,j\}} & \xrightarrow{\quad\quad\quad} & Y_{\{i,j\}} \\ F_{\{i,\dots,j\}} \downarrow & & \downarrow F_{\{i,j\}} \\ X_{\{i,\dots,j\}} & \xrightarrow{\quad\quad\quad} & X_{\{i,j\}} \end{array} \tag{8}$$

is homotopy Cartesian, and

- (2) the simplicial space Y_\bullet is 1-Segal.

Remark. By analogy with the difference between the left and right relative 1-Segal conditions, instead of the diagram (8) one could require that the outside square of

the diagram

$$\begin{array}{ccccc}
 Y_n & \longrightarrow & Y_{\{0,\dots,i,j,\dots,n\}} & \xrightarrow{F_{\{0,\dots,i,j,\dots,n\}}} & X_{\{0,\dots,i,j,\dots,n\}} \\
 \downarrow & & \downarrow & & \downarrow \\
 Y_{\{i,\dots,j\}} & \longrightarrow & Y_{\{i,j\}} & \xrightarrow{F_{\{i,j\}}} & X_{\{i,j\}}
 \end{array}$$

be homotopy Cartesian. However, morphisms F_\bullet satisfying these conditions seem to be less interesting. For example, such an F_\bullet does not lead in any obvious way to categorified Hall algebra representations.

The following result will be helpful in verifying the relative 2-Segal conditions.

Proposition 2.3. *Let $F_\bullet : Y_\bullet \rightarrow X_\bullet$ be a morphism of semi-simplicial spaces. The following statements are equivalent:*

- (1) *For every $n \geq 1$ the square (8) is homotopy Cartesian.*
- (2) *Y_\bullet is 2-Segal and for every $n \geq 1$ the outside square of the diagram (8) is homotopy Cartesian if $i = 0$ or $j = n$.*
- (3) *Y_\bullet is 2-Segal and for every $n \geq 1$ the square*

$$\begin{array}{ccc}
 Y_{\{0,\dots,n\}} & \longrightarrow & Y_{\{0,n\}} \\
 F_{\{0,\dots,n\}} \downarrow & & \downarrow F_{\{0,n\}} \\
 X_{\{0,\dots,n\}} & \longrightarrow & X_{\{0,n\}}
 \end{array} \tag{9}$$

is homotopy Cartesian.

Proof. Assume that the first condition holds. Since the bottom square of (8) is a degenerate version of the outer square, the 2-out-of-3 property of weak equivalences implies that Y_\bullet is 2-Segal. Hence the second condition holds. It is clear that the second condition implies the third. Assume that the third condition holds. Then the bottom square of diagram (8) is homotopy Cartesian by assumption while the top square is homotopy Cartesian since Y_\bullet is 2-Segal. The 2-out-of-3 property of weak equivalences now implies that the outside square of diagram (8) is homotopy Cartesian, showing that the first condition holds. \square

We will often show that a morphism $F_\bullet : Y_\bullet \rightarrow X_\bullet$ is relative 2-Segal by first proving that Y_\bullet is 1-Segal and then verifying the third condition of Proposition 2.3.

We briefly describe a model theoretic interpretation of relative 2-Segal spaces. Denote by \mathcal{R} the Reedy model structure on \mathbb{S}_Δ . Following [5, §5.2], let \mathbf{Seg}_2 be the left Bousfield localization of \mathcal{R} along the maps

$$\mathfrak{Seg}_2 = \{\Delta^\mathcal{T} \hookrightarrow \Delta^n \mid n \geq 3, \mathcal{T} \text{ is a triangulation of } P_n\}.$$

The fibrant objects of $(\mathbb{S}_\Delta, \mathbf{Seg}_2)$ are the (Reedy fibrant) 2-Segal combinatorial simplicial spaces. For each $X_\bullet \in \mathbb{S}_\Delta$ write $(\mathbf{Seg}_2)_{/X_\bullet}$ for the induced model structure on $(\mathbb{S}_\Delta)_{/X_\bullet}$ and let $(\mathbf{Seg}_2)_{/X_\bullet}^{\text{fib}}$ be its left Bousfield localization along the maps

$$\mathfrak{Seg}_2^{\text{fib}} = \{\Delta^{\{0,n\}} \hookrightarrow \Delta^n \xrightarrow{x_n} X_\bullet \mid x_n \in X_n, n \geq 2\}.$$

Assuming that X_\bullet is 2-Segal, Proposition 2.3 implies that the fibrant objects of $(\mathbf{Seg}_2)_{/X_\bullet}^{\text{fib}}$ are the morphisms $Y_\bullet \rightarrow X_\bullet$ for which the outside square of the diagrams (8) are homotopy Cartesian. From this point of view, a relative 2-Segal space is a fibrant object of $(\mathbf{Seg}_2)_{/X_\bullet}^{\text{fib}}$ whose total space is in addition 1-Segal. It is important to note that the main results of this paper do not hold for general fibrant objects of $(\mathbf{Seg}_2)_{/X_\bullet}^{\text{fib}}$.

The next definition requires that X_\bullet be a 2-Segal simplicial space.

Definition. A relative 2-Segal simplicial space $F_\bullet : Y_\bullet \rightarrow X_\bullet$ is called *unital relative 2-Segal* if for every $n \geq 2$ and every $0 \leq i \leq n-1$ the outside square of the diagram

$$\begin{array}{ccc} Y_{n-1} & \xrightarrow{s_i} & Y_n \\ \downarrow & & \downarrow \\ Y_{\{i\}} & \longrightarrow & Y_{\{i,i+1\}} \\ F_{\{i\}} \downarrow & & \downarrow F_{\{i,i+1\}} \\ X_{\{i\}} & \longrightarrow & X_{\{i,i+1\}} \end{array}$$

is homotopy Cartesian.

We have the following relative analogue of Proposition 1.1.

Proposition 2.4. Let $F_\bullet : Y_\bullet \rightarrow X_\bullet$ be a right relative 1-Segal semi-simplicial space. Then F_\bullet is relative 2-Segal. If in fact F_\bullet is a morphism of simplicial spaces, then F_\bullet is unital.

Proof. Proposition 2.1 implies that Y_\bullet is 1-Segal. The relative 2-Segal morphism

$$Y_{\{0,\dots,n\}} \rightarrow X_{\{0,\dots,n\}} \times_{X_{\{0,n\}}}^R Y_{\{0,n\}}$$

factors as the composition

$$\begin{aligned} Y_{\{0,\dots,n\}} &\rightarrow X_{\{0,\dots,n\}} \times_{X_{\{n\}}}^R Y_{\{n\}} \\ &\rightarrow X_{\{0,\dots,n\}} \times_{X_{\{0,n\}}}^R X_{\{0,n\}} \times_{X_{\{n\}}}^R Y_{\{n\}} \\ &\rightarrow X_{\{0,\dots,n\}} \times_{X_{\{0,n\}}}^R Y_{\{0,n\}}. \end{aligned}$$

The first and third morphisms are weak equivalences by the right relative 1-Segal conditions on F_\bullet , while the second morphism is a weak equivalence for trivial reasons. Hence the composition is a weak equivalence.

The unital condition is verified in a similar way. \square

Example. For any 1-Segal semi-simplicial space X_\bullet , the identity morphism $\mathbf{1}_{X_\bullet}$ is relative 2-Segal. However, if X_\bullet is only 2-Segal, then $\mathbf{1}_{X_\bullet}$ is not relative 2-Segal. \triangleleft

We end this section with a simple construction of relative 2-Segal spaces which can be seen as the 2-Segal analogue of the statement that the identity map between 1-Segal spaces is relative 1-Segal. Recall the left join functor

$$l : \Delta \rightarrow \Delta, \quad \{0, \dots, n\} \mapsto \{0', 0, \dots, n\}.$$

The left path space of a simplicial space X_\bullet is defined to be the composition

$$P^\triangleleft X_\bullet : \Delta^{\text{op}} \xrightarrow{l^{\text{op}}} \Delta^{\text{op}} \xrightarrow{X_\bullet} \text{Top}.$$

Identifying $P^\triangleleft X_m$ with X_{m+1} for each $m \geq 0$, the face map $\partial_i^\triangleleft : P^\triangleleft X_n \rightarrow P^\triangleleft X_{n-1}$ is equal to ∂_{i+1} . The remaining face maps $\partial_{0'}$ assemble to a simplicial morphism $F_\bullet^\triangleleft : P^\triangleleft X_\bullet \rightarrow X_\bullet$. Using instead the right join functor

$$r : \Delta \rightarrow \Delta, \quad \{0, \dots, n\} \mapsto \{0, \dots, n, n'\}$$

we obtain the right path space $F_\bullet^\triangleright : P^\triangleright X_\bullet \rightarrow X_\bullet$.

Proposition 2.5. Assume that X_\bullet is (unital) 2-Segal. Then the left and right path spaces

$$F_\bullet^\triangleleft : P^\triangleleft X_\bullet \rightarrow X_\bullet, \quad F_\bullet^\triangleright : P^\triangleright X_\bullet \rightarrow X_\bullet$$

are (unital) relative 2-Segal.

Proof. Since X_\bullet is 2-Segal, the path space criterion [5, Theorem 6.3.2], [10, Theorem 4.11] implies that $P^\triangleleft X_\bullet$ and $P^\triangleright X_\bullet$ are 1-Segal. The relative 2-Segal map

$$P^\triangleleft X_n = X_{\{0',0,\dots,n\}} \rightarrow X_n \times_{X_{\{0,n\}}}^R X_{\{0',0,n\}} = X_n \times_{X_{\{0,n\}}}^R P^\triangleleft X_{\{0,n\}}$$

and the relative unit map

$$P^\triangleleft X_{n-1} = X_{\{0',0,\dots,n-1\}} \rightarrow X_{\{i\}} \times_{X_{\{i,i+1\}}}^R X_{\{0',0,\dots,n\}} = X_{\{i\}} \times_{X_{\{i,i+1\}}}^R P^\triangleleft X_n$$

are 2-Segal and unit maps for X_\bullet , respectively, and so are weak equivalences by assumption. A similar calculation applies to right path spaces. \square

2.3. Symmetric polyhedral subdivisions. In this section we use combinatorial geometry to formulate the relative 2-Segal conditions on a morphism $F_\bullet : Y_\bullet \rightarrow X_\bullet$. This allows for a uniform treatment of the two conditions which define relative 2-Segal spaces.

For each $n \geq 0$ let \mathbf{n} be the ordered set $\{0 < \dots < n < n' < \dots < 0'\}$. There is a unique isomorphism $\mathbf{n} \simeq [2n+1]$ in Δ . Let $P_\mathbf{n} \subset \mathbb{R}^2$ be a convex $(2n+2)$ -gon which is symmetric with respect to reflection about the y -axis and which has no vertices on the y -axis. The vertices of $P_\mathbf{n}$ and \mathbf{n} can then be identified in a way which is consistent with the counterclockwise orientation of \mathbb{R}^2 and so that reflection about the y -axis interchanges i and i' . A polyhedral subdivision \mathcal{P} of $P_\mathbf{n}$ is called symmetric if it is fixed by reflection about the y -axis. Explicitly, \mathcal{P} is symmetric precisely if it consists of

- (i) horizontal diagonals $\{i', i\}$, for some $0 \leq i \leq n$, and
- (ii) pairs of diagonals $\{i, j\}$ and $\{i', j'\}$, for some $0 \leq i < j \leq n$,

and these diagonals have no intersection on the two dimensional interior of $P_\mathbf{n}$.

Definition. The category $\mathcal{S}_\mathcal{P}$ of symmetric simplices in \mathcal{P} is defined as follows.

- Objects of $\mathcal{S}_\mathcal{P}$ are \mathbb{Z}_2 -equivariant monomorphisms of simplicial sets $\sigma : D \hookrightarrow \Delta^\mathcal{P}$ of the form

$$\Delta^k \sqcup \Delta^k \hookrightarrow \Delta^\mathcal{P} \quad \text{or} \quad \Delta^\mathbf{m} \hookrightarrow \Delta^\mathcal{P},$$

where \mathbb{Z}_2 acts on $\Delta^k \sqcup \Delta^k$ by interchanging the two copies of Δ^k and acts on $\Delta^\mathbf{m}$ by first interchanging i and i' and then using the canonical isomorphism $\mathbf{m}^{\text{op}} \simeq \mathbf{m}$.

- Morphisms in $\mathcal{S}_\mathcal{P}$ are commutative diagrams of simplicial sets

$$\begin{array}{ccc} D & \xrightarrow{\phi} & D' \\ & \searrow & \swarrow \\ & \Delta^\mathcal{P} & \end{array}$$

It follows that a morphism ϕ in $\mathcal{S}_\mathcal{P}$ is of one of the following three types:

- (i) $\Delta^k \sqcup \Delta^k \hookrightarrow \Delta^l \sqcup \Delta^l$ for some $k \leq l$, in which case ϕ is determined by its restriction $\Delta^k \hookrightarrow \Delta^l$ to a summand.
- (ii) $\Delta^\mathbf{m} \hookrightarrow \Delta^\mathbf{n}$ for some $m \leq n$, in which case ϕ is determined by its restriction $\Delta^m \hookrightarrow \Delta^n$ for the canonical inclusions $[m] \hookrightarrow \mathbf{m}$ and $[n] \hookrightarrow \mathbf{n}$.
- (iii) $\Delta^k \sqcup \Delta^k \hookrightarrow \Delta^\mathbf{m}$ for some $k \leq m$, in which case ϕ is determined by its restriction $\Delta^k \hookrightarrow \Delta^m$.

Given a morphism of semi-simplicial spaces $F_\bullet : Y_\bullet \rightarrow X_\bullet$ and a symmetric polyhedral subdivision \mathcal{P} of $P_\mathbf{n}$, define a functor $F_\mathcal{P} : \mathcal{S}_\mathcal{P}^{\text{op}} \rightarrow \mathbf{Top}$ as follows. At the level of objects $F_\mathcal{P}$ is given by

$$F_\mathcal{P}(\Delta^k \sqcup \Delta^k \hookrightarrow \Delta^\mathcal{P}) = X_k, \quad F_\mathcal{P}(\Delta^\mathbf{m} \hookrightarrow \Delta^\mathcal{P}) = Y_m.$$

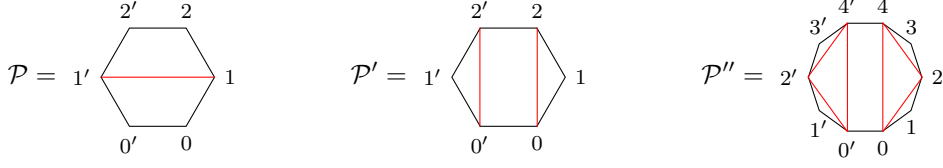


FIGURE 1. Examples of symmetric polyhedral subdivisions.

Applied to a morphism ϕ of type (i), (ii) or (iii), the morphism $F_{\mathcal{P}}(\phi)$ is defined in the obvious way using the semi-simplicial structure of X_{\bullet} , the semi-simplicial structure of Y_{\bullet} and the morphism F_{\bullet} , respectively. The symmetric membrane space of F_{\bullet} is then defined to be the homotopy limit

$$(\Delta^{\mathcal{P}}, F_{\bullet})_R = \operatorname{holim}_{\sigma \in \mathcal{S}_{\mathcal{P}}}^{\operatorname{Top}} F_{\mathcal{P}}(\sigma).$$

A refinement \mathcal{P} of \mathcal{P}' defines a simplicial subset $\Delta^{\mathcal{P}} \subset \Delta^{\mathcal{P}'}$ and thus a canonical morphism $(\Delta^{\mathcal{P}'}, F_{\bullet})_R \rightarrow (\Delta^{\mathcal{P}}, F_{\bullet})_R$. In particular, taking \mathcal{P}' to be the trivial subdivision and noting that $(\Delta^{\mathbf{n}}, F_{\bullet})_R \simeq Y_n$, we obtain a morphism

$$f_{\mathcal{P}}^{F_{\bullet}} : Y_n \rightarrow (\Delta^{\mathcal{P}}, F_{\bullet})_R.$$

Example. Consider the symmetric polyhedral subdivisions \mathcal{P} and \mathcal{P}' of P_2 depicted in Figure 1. The associated symmetric membrane spaces are homotopy fibre products,

$$(\Delta^{\mathcal{P}}, F_{\bullet})_R \simeq Y_{\{0,1\}} \times_{Y_{\{1\}}}^R Y_{\{1,2\}}, \quad (\Delta^{\mathcal{P}'}, F_{\bullet})_R \simeq X_{\{0,1,2\}} \times_{X_{\{0,2\}}}^R Y_{\{0,2\}}.$$

For more complicated subdivisions, such as \mathcal{P}'' of the same figure, a limit is indeed required to define the associated symmetric membrane space. \triangleleft

Proposition 2.6. *Let $F_{\bullet} : Y_{\bullet} \rightarrow X_{\bullet}$ be a morphism of semi-simplicial spaces. Assume that X_{\bullet} is 2-Segal. Then the following statements are equivalent:*

- (1) *The morphism F_{\bullet} is relative 2-Segal.*
- (2) *For every $n \geq 2$ and every symmetric polyhedral subdivision \mathcal{P} of P_n the morphism $f_{\mathcal{P}}^{F_{\bullet}}$ is a weak equivalence.*
- (3) *For every $n \geq 2$ and every maximal symmetric polyhedral subdivision \mathcal{P} of P_n the morphism $f_{\mathcal{P}}^{F_{\bullet}}$ is a weak equivalence.*

Proof. Assume that the second statement holds. If \mathcal{P} is the symmetric polyhedral subdivision of P_n consisting only of the diagonal $\{i', i\}$, then

$$(\Delta^{\mathcal{P}}, F_{\bullet})_R \simeq Y_{\{0, \dots, i\}} \times_{Y_{\{i\}}}^R Y_{\{i, \dots, n\}}$$

while if \mathcal{P}' consists only of the diagonals $\{0, n\}$ and $\{0', n'\}$, then

$$(\Delta^{\mathcal{P}'}, F_{\bullet})_R \simeq X_{\{0, \dots, n\}} \times_{X_{\{0, n\}}}^R Y_{\{0, n\}}.$$

Hence $f_{\mathcal{P}}^{F_{\bullet}}$ and $f_{\mathcal{P}'}^{F_{\bullet}}$ reduce to 1-Segal maps for Y_{\bullet} and relative 2-Segal maps for F_{\bullet} , respectively, and F_{\bullet} is relative 2-Segal.

Conversely, assume that F_{\bullet} is relative 2-Segal. Given a symmetric polyhedral subdivision \mathcal{P} , we can choose a sequence of symmetric polyhedral subdivisions $\mathcal{P}_0, \mathcal{P}_1, \dots, \mathcal{P}_k = \mathcal{P}$ such that \mathcal{P}_0 is the trivial subdivision and \mathcal{P}_j is obtained from \mathcal{P}_{j-1} by adding a single horizontal diagonal or a mirror pair of non-symmetric diagonals. Then $f_{\mathcal{P}}^{F_{\bullet}}$ is equal to the composition

$$Y_n \simeq (\Delta^{\mathcal{P}_0}, F_{\bullet})_R \rightarrow (\Delta^{\mathcal{P}_1}, F_{\bullet})_R \rightarrow \dots \rightarrow (\Delta^{\mathcal{P}_{k-1}}, F_{\bullet})_R \rightarrow (\Delta^{\mathcal{P}}, F_{\bullet})_R.$$

The construction of the \mathcal{P}_j ensures that each morphism in this composition is a weak equivalence, being induced by a 1-Segal map for Y_\bullet or a relative 2-Segal map for F_\bullet . It follows that $f_{\mathcal{P}}^{F_\bullet}$ is a weak equivalence and the second statement holds.

The equivalence of the second and third statements is proved similarly. \square

2.4. Stable framed objects. We modify the Waldhausen \mathcal{S}_\bullet -construction so as to construct relative 2-Segal groupoids from an abelian category together with a choice of stability condition. This section is motivated by the Hall algebra representations from [38], [9].

Fix a field k . Let \mathcal{C} be a k -linear abelian category with Grothendieck group $K_0(\mathcal{C})$. A stability function on \mathcal{C} is a group homomorphism $Z : K_0(\mathcal{C}) \rightarrow \mathbb{C}$ such that

$$Z(A) \in \{me^{\pi i \phi} \mid m \in \mathbb{R}_{>0}, \phi \in (0, 1]\} \subset \mathbb{C}$$

for all non-zero objects $A \in \mathcal{C}$ [2, §2]. Let $\phi(A) \in (0, 1]$ be the phase of $Z(A)$. A non-zero object $A \in \mathcal{C}$ is called Z -semistable if $\phi(A') \leq \phi(A)$ for all non-trivial subobjects $A' \subset A$. For each $\phi \in (0, 1]$ the full subcategory $\mathcal{C}_\phi^{Z\text{-ss}} \subset \mathcal{C}$ consisting of the zero object together with all objects which are Z -semistable of phase ϕ is abelian. The Waldhausen space $BS_\bullet(\mathcal{C}_\phi^{Z\text{-ss}})$ is therefore 2-Segal by Theorem 1.2.

We formulate the notion of framing following [38, §4]. Fix a left exact functor $\Phi : \mathcal{C} \rightarrow \mathbf{Vect}_k$. A framed object of \mathcal{C} is then a pair (M, s) consisting of an object $M \in \mathcal{C}$ and a section $s \in \Phi(M)$. A morphism of framed objects $(M, s) \rightarrow (M', s')$ is a pair $(\pi, \lambda) \in \text{Hom}_{\mathcal{C}}(M, M') \times k$ which satisfies $\Phi(\pi)(s) = \lambda s'$. A framed object (M, s) is called stable framed if M is Z -semistable and $\phi(A) < \phi(M)$ for all proper subobjects $A \subset M$ for which $s \in \Phi(A) \subset \Phi(M)$.

Example.

- (1) Let \mathcal{C} be the category of finite dimensional representations of a quiver Q . For any $d \in \mathbb{Z}_{\geq 0}^{Q_0}$ the functor $\Phi : U \mapsto \bigoplus_{i \in Q_0} \text{Hom}_k(k^{d_i}, U_i)$ is a framing.
- (2) Let \mathcal{C} be the category of coherent sheaves on a smooth projective variety X . Then the global sections functor $\Phi = H^0(X, -)$ is a framing. \triangleleft

For each $n \geq 0$ let $\mathcal{S}_n^{\text{st-fr}}(\mathcal{C}_\phi^{Z\text{-ss}})$ be the maximal groupoid of the category of diagrams of the form

$$\begin{array}{ccccccc}
 A_{\{0,0\}} & \longrightarrow & A_{\{0,1\}} & \longrightarrow & \cdots & \longrightarrow & A_{\{0,n\}} & \longrightarrow & (M_0, s_0) \\
 & & \downarrow & & & & \downarrow & & \downarrow \\
 & & A_{\{1,1\}} & \longrightarrow & \cdots & \longrightarrow & A_{\{1,n\}} & \longrightarrow & (M_1, s_1) \\
 & & & & \ddots & & \downarrow & & \downarrow \\
 & & & & & & \vdots & & \vdots \\
 & & & & & & \downarrow & & \downarrow \\
 & & & & & & A_{\{n,n\}} & \longrightarrow & (M_n, s_n) \\
 & & & & & & & & \downarrow \\
 & & & & & & & & (M_{n+1}, s_{n+1})
 \end{array}$$

which have the following properties:

- (i) Upon forgetting the framing data s_0, \dots, s_{n+1} the resulting diagram is an object of $\mathcal{S}_{n+1}(\mathcal{C}_\phi^{Z\text{-ss}})$.
- (ii) Each pair (M_i, s_i) , $i = 0, \dots, n$, is a stable framed object.

In the above diagram a morphism $A_{\{i,n\}} \rightarrow (M_i, s_i)$ is simply a morphism of the underlying objects of \mathcal{C} . The groupoids $\mathcal{S}_n^{\text{st-fr}}(\mathcal{C}_\phi^{Z\text{-ss}})$ assemble to a simplicial groupoid. There is a canonical simplicial morphism $F_\bullet : \mathcal{S}_\bullet^{\text{st-fr}}(\mathcal{C}_\phi^{Z\text{-ss}}) \rightarrow \mathcal{S}_\bullet(\mathcal{C}_\phi^{Z\text{-ss}})$ which forgets the rightmost column and bottom row of a diagram.

Theorem 2.7. *Let \mathcal{C} be an abelian category with stability function Z and framing Φ . For each $\phi \in (0, 1]$ the map $BF_\bullet : BS_\bullet^{\text{st-fr}}(\mathcal{C}_\phi^{Z\text{-ss}}) \rightarrow BS_\bullet(\mathcal{C}_\phi^{Z\text{-ss}})$ is relative 2-Segal.*

Proof. The proof that $BS_\bullet^{\text{st-fr}}(\mathcal{C}_\phi^{Z\text{-ss}})$ is 1-Segal reduces to the 2-Segal property of $BS_\bullet(\mathcal{C}_\phi^{Z\text{-ss}})$, so we omit it. To verify the second of the relative 2-Segal conditions it suffices to show that the functor

$$\Psi_n : \mathcal{S}_n^{\text{st-fr}}(\mathcal{C}_\phi^{Z\text{-ss}}) \rightarrow \mathcal{S}_n(\mathcal{C}_\phi^{Z\text{-ss}}) \times_{\mathcal{S}_{\{0,n\}}(\mathcal{C}_\phi^{Z\text{-ss}})}^{(2)} \mathcal{S}_{\{0,n\}}^{\text{st-fr}}(\mathcal{C}_\phi^{Z\text{-ss}})$$

is an equivalence. Here $- \times^{(2)} -$ denotes the 2-pullback of groupoids.

For each $n \geq 0$ let $\mathcal{F}_n(\mathcal{C}_\phi^{Z\text{-ss}})$ be the groupoid of n -flags in $\mathcal{C}_\phi^{Z\text{-ss}}$, that is, diagrams in $\mathcal{C}_\phi^{Z\text{-ss}}$ of the form

$$0 \rightharpoonup A_1 \rightharpoonup \cdots \rightharpoonup A_{n-1} \rightharpoonup A_n.$$

The forgetful functor $\mu_n : \mathcal{S}_n(\mathcal{C}_\phi^{Z\text{-ss}}) \rightarrow \mathcal{F}_n(\mathcal{C}_\phi^{Z\text{-ss}})$ is an equivalence of groupoids [43, §1.3]. Similarly, let $\mathcal{F}_n^{\text{st-fr}}(\mathcal{C}_\phi^{Z\text{-ss}})$ be the groupoid of n -flags in a stable framed object. An object of $\mathcal{F}_n^{\text{st-fr}}(\mathcal{C}_\phi^{Z\text{-ss}})$ is a diagram of the form

$$0 \rightharpoonup A_1 \rightharpoonup \cdots \rightharpoonup A_{n-1} \rightharpoonup A_n \rightharpoonup (M, s) \quad (10)$$

with $A_1, \dots, A_n, M \in \mathcal{C}_\phi^{Z\text{-ss}}$ and (M, s) stable framed. We claim that the forgetful functor $\nu_n : \mathcal{S}_n^{\text{st-fr}}(\mathcal{C}_\phi^{Z\text{-ss}}) \rightarrow \mathcal{F}_n^{\text{st-fr}}(\mathcal{C}_\phi^{Z\text{-ss}})$ is an equivalence. A quasi-inverse η_n of ν_n can be constructed as follows. Given a flag (10) set $A_{\{0,k\}} = A_k$, $1 \leq k \leq n$, and $(M_0, s_0) = (M, s)$. Define $A_{\{1,k\}}$, $1 \leq k \leq n$, and M_1 as the pushouts

$$\begin{array}{ccc} A_{\{0,1\}} & \rightharpoonup & A_{\{0,k\}} \\ \downarrow & & \downarrow \\ 0 & \rightharpoonup & A_{\{1,k\}} \end{array} \quad \begin{array}{ccc} A_{\{0,1\}} & \rightharpoonup & M_0 \\ \downarrow & & \downarrow \\ 0 & \rightharpoonup & M_1 \end{array}$$

Since $\mathcal{C}_\phi^{Z\text{-ss}}$ is abelian, it is automatic that each $A_{\{1,k\}}$ and M_1 are Z -semistable of phase ϕ . Let s_1 be the image of s_0 under the morphism $\Phi(M_0) \rightarrow \Phi(M_1)$. We need to show that (M_1, s_1) is stable framed. Note that s_1 is non-zero. Indeed, if $s_1 = 0$, then $s_0 \in \Phi(A_{\{0,1\}}) \subset \Phi(M_0)$ and $\phi(A_{\{0,1\}}) = \phi(M_0)$, contradicting the assumed framed stability of (M_0, s_0) . Let then $B_1 \subset M_1$ be a proper subobject with $s_1 \in \Phi(B_1) \subset \Phi(M_1)$. There exists a unique object $B_0 \in \mathcal{C}$ which fits into the exact commutative diagram

$$\begin{array}{ccccc} A_{\{0,1\}} & \rightharpoonup & B_0 & \longrightarrow & B_1 \\ \downarrow \text{id} & & \downarrow & & \downarrow \\ A_{\{0,1\}} & \rightharpoonup & M_0 & \longrightarrow & M_1 \\ & & \downarrow & & \downarrow \\ & & M_0/B_0 & \xrightarrow{\sim} & M_1/B_1 \end{array}$$

Applying Φ gives a commutative diagram of vector spaces

$$\begin{array}{ccccc}
 \Phi(A_{\{0,1\}}) & \hookrightarrow & \Phi(B_0) & \longrightarrow & \Phi(B_1) \\
 \downarrow \text{id} & & \downarrow & & \downarrow \\
 \Phi(A_{\{0,1\}}) & \hookrightarrow & \Phi(M_0) & \longrightarrow & \Phi(M_1) \\
 & & \downarrow & & \downarrow \\
 & & \Phi(M_0/B_0) & \xrightarrow{\sim} & \Phi(M_1/B_1)
 \end{array}$$

Since $s_1 \in \Phi(B_1)$ we see that s_1 is in the kernel of $\Phi(M_1) \rightarrow \Phi(M_1/B_1)$. It follows that s_0 is in the kernel of $\Phi(M_0) \rightarrow \Phi(M_0/B_0)$. Hence $s_0 \in \Phi(B_0) \subset \Phi(M_0)$. Since (M_0, s_0) is stable framed we have $\phi(B_0) < \phi(M_0)$. The equalities $\phi(A_{\{0,1\}}) = \phi(M_0) = \phi(M_1)$ combined with the standard See-Saw properties of stability functions then give $\phi(M_1) > \phi(B_1)$, as desired. This procedure defines the top two rows of $\eta_n(A_{\{0,\bullet\}} \rightarrow (M_0, s_0))$ and can be iterated to define the remaining $n - 2$ rows. This defines the desired quasi-inverse.

To complete the proof, consider the following commutative diagram of functors:

$$\begin{array}{ccc}
 \mathcal{S}_n^{\text{st-fr}}(\mathcal{C}_\phi^{Z\text{-ss}}) & \xrightarrow{\Psi_n} & \mathcal{S}_n(\mathcal{C}_\phi^{Z\text{-ss}}) \times_{\mathcal{S}_{\{0,n\}}(\mathcal{C}_\phi^{Z\text{-ss}})}^{(2)} \mathcal{S}_{\{0,n\}}^{\text{st-fr}}(\mathcal{C}_\phi^{Z\text{-ss}}) \\
 \downarrow \nu_n & & \downarrow \mu_n \times \nu_{\{0,n\}} \\
 \mathcal{F}_n^{\text{st-fr}}(\mathcal{C}_\phi^{Z\text{-ss}}) & \xrightarrow[\tilde{\Psi}_n]{} & \mathcal{F}_n(\mathcal{C}_\phi^{Z\text{-ss}}) \times_{\mathcal{F}_{\{0,n\}}(\mathcal{C}_\phi^{Z\text{-ss}})}^{(2)} \mathcal{F}_{\{0,n\}}^{\text{st-fr}}(\mathcal{C}_\phi^{Z\text{-ss}})
 \end{array}$$

The discussion above shows that the vertical functors are equivalences; note that $\mathcal{S}_1(\mathcal{C}_\phi^{Z\text{-ss}})$ and $\mathcal{F}_1(\mathcal{C}_\phi^{Z\text{-ss}})$ are canonically equivalent. That Ψ_n is an equivalence therefore reduces to the statement that $\tilde{\Psi}_n$ is an equivalence, which is obvious. \square

2.5. Real pseudo-holomorphic polygons. Recall that an almost complex structure on a smooth manifold M is an endomorphism $J : TM \rightarrow TM$ of the tangent bundle which satisfies $J \circ J = -\mathbf{1}_{TM}$. Given almost complex manifolds (Σ, j) and (M, J) , a continuous map $u : \Sigma \rightarrow M$ is called pseudo-holomorphic if it is smooth and satisfies the equation

$$du + J \circ du \circ j = 0.$$

Let \mathbb{H} be the Lobachevsky plane, realized as the open unit disk in \mathbb{C} with centre the origin and metric

$$ds^2 = \frac{dz \wedge d\bar{z}}{(1 - |z|^2)^2}.$$

The ideal boundary $\partial\mathbb{H}$ of \mathbb{H} is the unit circle in \mathbb{C} . Denote by $j_{\mathbb{H}}$ the complex structure of \mathbb{H} . The group $\text{SL}_2(\mathbb{R})$ acts on \mathbb{H} by orientation preserving isometries.

We require some basic definitions from Teichmüller theory [26]. Given $b, b' \in \partial\mathbb{H}$, let (b, b') be the oriented geodesic with limiting points b at infinite negative time and b' at infinite positive time. For each $n \geq 0$ let $P(b_0, \dots, b_n)$ be the ideal $(n+1)$ -gon in \mathbb{H} with vertices $b_0, \dots, b_n \in \partial\mathbb{H}$ numbered compatibly with the canonical orientation of $\partial\mathbb{H}$. A decoration of $P(b_0, \dots, b_n)$ is the data of a horocycle $\xi_i \in \text{Hor}_{b_i}$ with hyperbolic centre b_i for each $i = 0, \dots, n$. The group $\text{SL}_2(\mathbb{R}) \times \mathbb{R}$ acts on the set of decorated ideal $(n+1)$ -gons by the formula

$$(g, a) \cdot (P(b_0, \dots, b_n), \xi_0, \dots, \xi_n) = (P(g \cdot b_0, \dots, g \cdot b_n), g \cdot \xi_0 + a, \dots, g \cdot \xi_n + a)$$

where we have used the canonical identification of \mathbb{R} -torsors $g : \text{Hor}_{b_i} \rightarrow \text{Hor}_{g \cdot b_i}$. We will often omit the decoration from the notation if it will not cause confusion.

Given a decorated ideal $(n+1)$ -gon, for distinct indices $0 \leq i < j \leq n$ let $m(\xi_i, \xi_j)$ be the midpoint of the unique geodesic connecting the points $\xi_i \cap (b_i, b_j)$ and $\xi_j \cap (b_i, b_j)$. This trivializes the \mathbb{R} -torsor (b_i, b_j) via

$$[-\infty, \infty] \rightarrow (b_i, b_j), \quad 0 \mapsto m(\xi_i, \xi_j). \quad (11)$$

Note that this trivialization is invariant under the above action of \mathbb{R} .

Following [5, §3.8] we construct from an almost complex manifold (M, J) a semi-simplicial set $\tilde{\mathbb{T}}_\bullet(M)$. Let $\tilde{\mathbb{T}}_0(M) = M$ and, for each $n \geq 1$, let $\tilde{\mathbb{T}}_n(M)$ be the set of equivalence classes of pairs consisting of a decorated ideal $(n+1)$ -gon P together with a continuous map $u : (P, j_{\mathbb{H}}) \rightarrow (M, J)$ which is pseudo-holomorphic on the two dimensional interior of P . The equivalence relation is generated by the action of $\mathrm{SL}_2(\mathbb{R}) \times \mathbb{R}$ on decorated ideal polygons. By using the trivialization (11) we identify $\tilde{\mathbb{T}}_1(M)$ with the set of continuous maps $[-\infty, \infty] \rightarrow M$. The face map $\partial_i : \tilde{\mathbb{T}}_n(M) \rightarrow \tilde{\mathbb{T}}_{n-1}(M)$ omits the ideal boundary point b_i and forms the resulting decorated n -gon together with the restricted morphism to M . It is proved in [5, Theorem 3.8.6] that $\tilde{\mathbb{T}}_\bullet(M)$ is a 2-Segal semi-simplicial set.

Turning to the relative setting, recall that a real structure on (M, J) is a smooth map $\tau : M \rightarrow M$ which satisfies $\tau \circ \tau = \mathbf{1}_M$ and

$$d\tau - J \circ d\tau \circ J = 0.$$

A map $u : (\Sigma, j, \sigma) \rightarrow (M, J, \tau)$ of almost complex manifolds with real structures is called real pseudo-holomorphic if it is pseudo-holomorphic and satisfies $u \circ \sigma = \tau \circ u$.

The real structure $\sigma : z \mapsto -\bar{z}$ on \mathbb{C} induces a real structure on \mathbb{H} , again denoted by σ . The subgroup $\mathrm{SL}_2(\mathbb{R})^\sigma \leq \mathrm{SL}_2(\mathbb{R})$ which commutes with σ is isomorphic to $\mathbb{R}^\times \rtimes \mathbb{Z}_2$. The generator of \mathbb{Z}_2 is rotation of \mathbb{H} through an angle π . Define a σ -real ideal boundary point of \mathbb{H} to be either a point of the real locus $\partial\mathbb{H}^\sigma = \{\sqrt{-1}, -\sqrt{-1}\}$ or a pair $\{b, \sigma(b)\}$ of distinct σ -conjugate ideal boundary points.

Fix $n \geq 0$. A real ideal $(n+1)$ -gon $Q = Q(b_0, \dots, b_n, \dots)$ is an ideal polygon which has exactly $n+1$ σ -real ideal boundary points, labelled so that only b_0 and b_n may lie in $\partial\mathbb{H}^\sigma$. It follows that Q is of one of the following four types:

- (i) $Q(b_0, \dots, b_n, \sigma(b_n), \sigma(b_{n-1}), \dots, \sigma(b_1), \sigma(b_0))$
- (ii) $Q(b_0, \dots, b_n, \sigma(b_n), \sigma(b_{n-1}), \dots, \sigma(b_1))$
- (iii) $Q(b_0, \dots, b_n, \sigma(b_{n-1}), \dots, \sigma(b_1), \sigma(b_0))$
- (iv) $Q(b_0, \dots, b_n, \sigma(b_{n-1}), \dots, \sigma(b_1))$.

Then Q has exactly zero, one, one and two ideal vertices in $\partial\mathbb{H}^\sigma$ in cases (i)-(iv), respectively. A decoration of Q is a decoration of its underlying ideal polygon for which the horocycle at a vertex b is equal to that at $\sigma(b)$ under the canonical identification $\mathrm{Hor}_b \simeq \mathrm{Hor}_{\sigma(b)}$. The group $\mathrm{SL}_2(\mathbb{R})^\sigma \times \mathbb{R}$ acts on the set of real decorated ideal polygons.

With the above notation in place, we define a semi-simplicial set $\tilde{\mathbb{T}}_\bullet^\tau(M)$ as follows. For each $n \geq 0$ let $\tilde{\mathbb{T}}_n^\tau(M)$ be the set of equivalence classes of pairs (Q, v) consisting of a real decorated ideal $(n+1)$ -gon Q together with real continuous map $v : (Q, j_{\mathbb{H}}, \sigma) \rightarrow (M, J, \tau)$ which is pseudo-holomorphic on the two dimensional interior of Q . The equivalence relation is generated by the action of $\mathrm{SL}_2(\mathbb{R})^\sigma \times \mathbb{R}$ on real decorated ideal polygons. In particular, we have

$$\tilde{\mathbb{T}}_0^\tau(M) \simeq M^\tau \sqcup C^0([-\infty, \infty], M)^\tau \quad (12)$$

where $C^0([-\infty, \infty], M)^\tau$ denotes the set of real continuous maps $[-\infty, \infty] \rightarrow M$ with $[-\infty, \infty]$ given the \mathbb{Z}_2 -action $x \mapsto -x$. Defining face maps $\partial_i : \tilde{\mathbb{T}}_n^\tau(M) \rightarrow \tilde{\mathbb{T}}_{n-1}^\tau(M)$ by omitting the σ -real ideal boundary point $\{b_i, \sigma(b_i)\}$ defines a semi-simplicial set $\tilde{\mathbb{T}}_\bullet^\tau(M)$.

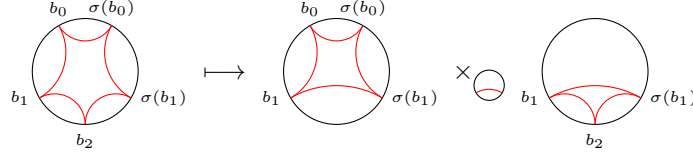


FIGURE 2. The map $\tilde{\mathbb{T}}_2^\tau(M) \rightarrow \tilde{\mathbb{T}}_{\{0,1\}}^\tau(M) \times_{\tilde{\mathbb{T}}_{\{1\}}^\tau(M)} \tilde{\mathbb{T}}_{\{1,2\}}^\tau(M)$, omitting the data of the maps to M .

Given $(Q, v) \in \tilde{\mathbb{T}}_n^\tau(M)$ write $Q = Q(b_0, \dots, b_n, \dots)$ and let $\tilde{Q} = \tilde{Q}(b_0, \dots, b_n)$ be the decorated ideal $(n+1)$ -gon obtained from Q by omitting the remaining vertices and their decorations. This defines a semi-simplicial morphism

$$F_\bullet : \tilde{\mathbb{T}}_\bullet^\tau(M) \rightarrow \tilde{\mathbb{T}}_\bullet(M), \quad (Q, v) \mapsto (\tilde{Q}, v|_{\tilde{Q}}).$$

We can now state the main result of this section.

Theorem 2.8. *For any almost complex manifold with real structure (M, J, τ) , the morphism $F_\bullet : \tilde{\mathbb{T}}_\bullet^\tau(M) \rightarrow \tilde{\mathbb{T}}_\bullet(M)$ is a relative 2-Segal semi-simplicial set.*

Proof. For each $n \geq 2$ and $0 < i < n$ we construct an inverse of the 1-Segal map

$$\Xi_n : \tilde{\mathbb{T}}_n^\tau(M) \rightarrow \tilde{\mathbb{T}}_{\{0, \dots, i\}}^\tau(M) \times_{\tilde{\mathbb{T}}_{\{i\}}^\tau(M)} \tilde{\mathbb{T}}_{\{i, \dots, n\}}^\tau(M).$$

Let $(Q', v') \in \tilde{\mathbb{T}}_{\{0, \dots, i\}}^\tau(M)$ and $(Q'', v'') \in \tilde{\mathbb{T}}_{\{i, \dots, n\}}^\tau(M)$ with equal image in $\tilde{\mathbb{T}}_{\{i\}}^\tau(M)$. Since $0 < i < n$ these images lie in the component $C^0([-\infty, \infty], M)^\tau$ with respect to the decomposition (12). There exists a unique $g \in \mathrm{SL}_2(\mathbb{R})^\sigma$ such that $g \cdot (b'_i, \sigma(b'_i)) = (b''_i, \sigma(b''_i))$ and the restrictions

$$g \cdot v', v'' : [-\infty, \infty] \rightarrow M$$

are equal. Applying Morera's theorem, we conclude that $g \cdot (Q', v')$ and (Q'', v'') can be glued in a unique way so as to obtain a continuous map

$$g \cdot v' \cup v'' : g \cdot Q' \cup_{(b''_i, \sigma(b''_i))} Q'' \rightarrow M$$

which is pseudo-holomorphic on the interior. Since both Q' and Q'' (resp. v' and v'') are real, so too is $g \cdot Q' \cup_{(b''_i, \sigma(b''_i))} Q''$ (resp. $g \cdot v' \cup v''$). See Figure 2. This defines an inverse of Ξ_n , showing that $\tilde{\mathbb{T}}_\bullet^\tau(M)$ is 1-Segal.

To verify the relative 2-Segal condition, for each $n \geq 1$ consider the map

$$\Psi_n : \tilde{\mathbb{T}}_n^\tau(M) \rightarrow \tilde{\mathbb{T}}_n(M) \times_{\tilde{\mathbb{T}}_{\{0, n\}}(M)} \tilde{\mathbb{T}}_{\{0, n\}}^\tau(M).$$

Let $(P', u') \in \tilde{\mathbb{T}}_n(M)$ and $(Q'', v'') \in \tilde{\mathbb{T}}_{\{0, n\}}^\tau(M)$ with equal image in $\tilde{\mathbb{T}}_{\{0, n\}}(M)$. Choose $g \in \mathrm{SL}_2(\mathbb{R})$ so that $g \cdot (b'_0, b'_n) = (b''_0, b''_n)$. As above, by using the reality condition on (Q'', v'') we conclude that, up to the action of $\mathrm{SL}_2(\mathbb{R})^\sigma$, the triple

$$\{g \cdot (P', u'), (Q'', v''), g \cdot (\sigma(P'), \tau \circ u' \circ \sigma^{-1})\}$$

can be glued in a unique way so as to obtain an element of $\tilde{\mathbb{T}}_n^\tau(M)$. See Figure 3. This defines the inverse of Ψ_n . \square

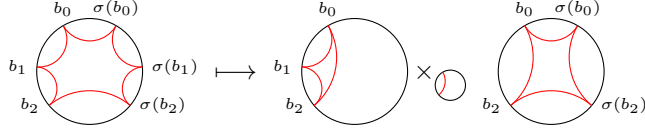


FIGURE 3. The map $\widetilde{\mathbb{T}}_2^{\tau}(M) \rightarrow \widetilde{\mathbb{T}}_2(M) \times_{\widetilde{\mathbb{T}}_{\{0,2\}}(M)} \widetilde{\mathbb{T}}_{\{0,2\}}^{\tau}(M)$, omitting the data of the maps to M .

3. RELATIVE SEGAL SPACES FROM CATEGORIES WITH DUALITIES

3.1. Categories with duality. For a detailed introduction to categories with duality the reader is referred to [32].

Definition. A category with (strong) duality is a triple (\mathcal{C}, P, Θ) consisting of a category \mathcal{C} , a contravariant functor $P : \mathcal{C} \rightarrow \mathcal{C}$ and a natural isomorphism $\Theta : \mathbf{1}_{\mathcal{C}} \rightarrow P \circ P$ such that $P(\Theta_U) \circ \Theta_{P(U)} = \mathbf{1}_{P(U)}$ for all objects $U \in \mathcal{C}$.

When there is no risk of confusion we will omit (P, Θ) from the notation and simply refer to \mathcal{C} as a category with duality.

A form functor $(T, \varphi) : (\mathcal{C}, P, \Theta) \rightarrow (\mathcal{D}, Q, \Xi)$ between categories with duality consists of a functor $T : \mathcal{C} \rightarrow \mathcal{D}$ and a natural transformation $\varphi : T \circ P \rightarrow Q \circ T$ for which the diagram

$$\begin{array}{ccc} T(U) & \xrightarrow{\Xi_{T(U)}} & Q^2 T(U) \\ T(\Theta_U) \downarrow & & \downarrow Q(\varphi_U) \\ TP^2(U) & \xrightarrow{\varphi_{P(U)}} & QTP(U) \end{array}$$

commutes for all objects $U \in \mathcal{C}$. In particular, a duality preserving functor $T : \mathcal{C} \rightarrow \mathcal{D}$, that is, a functor which satisfies $T \circ P = Q \circ T$ and $T(\Theta) = \Xi_T$, defines a form functor (T, id) . Write CatD for the category of small categories with duality with form functors as morphisms.

A (non-singular) symmetric form in \mathcal{C} is a pair (N, ψ_N) consisting of an object $N \in \mathcal{C}$ and an isomorphism $\psi_N : N \rightarrow P(N)$ which satisfies $P(\psi_N) \circ \Theta_N = \psi_N$. An isometry of symmetric forms $\phi : (M, \psi_M) \rightarrow (N, \psi_N)$ is an isomorphism $\phi : M \rightarrow N$ which satisfies $P(\phi) \circ \psi_N \circ \phi = \psi_M$. The groupoid of symmetric forms and their isometries is denoted by \mathcal{C}_h and is called the Hermitian groupoid of \mathcal{C} .

Recall that \mathbf{n} denotes the totally ordered set $\{0 < \dots < n < n' < \dots < 0'\}$. The morphisms $[n] \rightarrow \mathbf{n}$, $i \mapsto i$, define the subdivision functor $\text{sd} : \Delta \rightarrow \Delta$. The edgewise subdivision of a simplicial object X_{\bullet} of a category \mathcal{C} is then defined to be the composition

$$X_{\bullet}^e : \Delta^{\text{op}} \xrightarrow{\text{sd}^{\text{op}}} \Delta^{\text{op}} \xrightarrow{X_{\bullet}} \mathcal{C}.$$

Explicitly, $X_n^e = X_{\mathbf{n}} = X_{2n+1}$ with face maps

$$\partial_i^e : X_n^e = X_{2n+1} \xrightarrow{\partial_i \circ \partial_{2n+1-i}} X_{2n-1} = X_{n-1}^e.$$

The degeneracy maps admit a similar description. The morphisms $[n] \rightarrow \mathbf{n}$ define a morphism $X_{\bullet}^e \rightarrow X_{\bullet}$.

In the sections below we will repeatedly apply the following result (cf. [12, §1.5]).

Lemma 3.1. *Let X_\bullet be a simplicial object of \mathbf{Cat} . Let (P_n, Θ_n) , $n \geq 0$, be duality structures on X_n which satisfy*

$$P_{n-1} \circ \partial_i = \partial_{n-i} \circ P_n, \quad P_{n+1} \circ s_i = s_{n-i} \circ P_n$$

and

$$\partial_i(\Theta_n) = (\Theta_{n-1})_{\partial_i}, \quad s_i(\Theta_n) = (\Theta_{n+1})_{s_i}$$

for all $0 \leq i \leq n$. Then $(X_\bullet^e, P_\bullet^e, \Theta_\bullet^e)$ is a simplicial object of \mathbf{CatD} .

Proof. It is clear that $(X_n^e, P_n^e, \Theta_n^e) = (X_{2n+1}, P_{2n+1}, \Theta_{2n+1})$ is a category with duality. The equality $\partial_i(\Theta_n) = (\Theta_{n-1})_{\partial_i}$ together with the calculation

$$\begin{aligned} \partial_i^e \circ P_n^e &= \partial_i \circ \partial_{2n+1-i} \circ P_{2n+1} \\ &= \partial_i \circ P_{2n} \circ \partial_i \\ &= P_{2n-1} \circ \partial_{2n-i} \circ \partial_i \\ &= P_{2n-1} \circ \partial_i \circ \partial_{2n-1-i} = P_{n-1}^e \circ \partial_i^e. \end{aligned}$$

shows that the face map $(\partial_i^e, \text{id}) : X_n^e \rightarrow X_{n-1}^e$ is a form functor. A similar computation shows that $(s_i^e, \text{id}) : X_n^e \rightarrow X_{n+1}^e$ is a form functor. \square

3.2. Unoriented categorified nerves. As a warm-up for Section 3.3 we adapt the categorified nerve construction to the relative setting.

Let \mathcal{C} be a small category. Its categorified nerve $\mathcal{N}_\bullet(\mathcal{C})$ is a 1-Segal simplicial groupoid. Suppose that (P, Θ) is a duality structure on \mathcal{C} . Then $\mathcal{N}_n(\mathcal{C})$, $n \geq 0$, inherit duality structures which satisfy the hypothesis of Lemma 3.1. Hence $\mathcal{N}_\bullet^e(\mathcal{C})$ is a simplicial groupoid with duality. The categorified unoriented nerve $\mathcal{U}_\bullet(\mathcal{C})$ is defined to be $\mathcal{N}_\bullet^e(\mathcal{C})_h$. An object $(x_\bullet, \psi_\bullet) \in \mathcal{U}_n(\mathcal{C})$ is a diagram

$$x_0 \rightarrow \cdots \rightarrow x_n \rightarrow x_{n'} \rightarrow \cdots \rightarrow x_{0'}$$

in \mathcal{C} together with isomorphisms $\psi_i : x_i \rightarrow P(x_{i'})$, $0 \leq i \leq n'$, (with the convention that $i'' = i$) which satisfy $P(\psi_i) \circ \Theta_{x_{i'}} = \psi_{i'}$ and make the obvious squares commute. The forgetful morphism $F_\bullet : \mathcal{U}_\bullet(\mathcal{C}) \rightarrow \mathcal{N}_\bullet(\mathcal{C})$ is given by

$$\mathcal{U}_n(\mathcal{C}) \ni (x_\bullet, \psi_\bullet) \mapsto (x_0 \rightarrow \cdots \rightarrow x_n) \in \mathcal{N}_n(\mathcal{C}).$$

Proposition 3.2. *For any small category with duality (\mathcal{C}, P, Θ) , the morphism $BF_\bullet : B\mathcal{U}_\bullet(\mathcal{C}) \rightarrow B\mathcal{N}_\bullet(\mathcal{C})$ is right relative 1-Segal.*

Proof. We need to prove that, for each $0 < i < n$, the functor

$$\begin{aligned} \Psi_n : \mathcal{U}_n(\mathcal{C}) &\rightarrow \mathcal{N}_{\{0, \dots, i\}}(\mathcal{C}) \times_{\mathcal{N}_{\{i\}}(\mathcal{C})}^{(2)} \mathcal{U}_{\{i, \dots, n\}}(\mathcal{C}) \\ (x_\bullet, \psi_\bullet) &\mapsto (x_0 \rightarrow \cdots \rightarrow x_i, (x_i \rightarrow \cdots \rightarrow x_n \rightarrow x_{n'} \rightarrow \cdots \rightarrow x_{i'}, \psi_\bullet); 1) \end{aligned}$$

is an equivalence. An object of the above 2-pullback is a triple

$$(x_0 \rightarrow \cdots \rightarrow x_i, (x'_i \rightarrow \cdots \rightarrow x'_n \rightarrow x'_{n'} \rightarrow \cdots \rightarrow x'_{i'}, \psi_\bullet); \alpha) \quad (13)$$

with $\alpha : x_i \rightarrow x'_i$ an isomorphism. A morphism from the object (13) to a second, say

$$(y_0 \rightarrow \cdots \rightarrow y_i, (y'_i \rightarrow \cdots \rightarrow y'_n \rightarrow y'_{n'} \rightarrow \cdots \rightarrow y'_{i'}, \mu_\bullet); \beta),$$

is a collection $((p_0, \dots, p_i), (q_i, \dots, q_n, q_{n'}, \dots, q_{i'}))$ where $p_j : x_j \rightarrow y_j$, $1 \leq j \leq i$, and $q_j : x'_j \rightarrow y'_j$, $i \leq j \leq i'$, are isomorphisms which make the appropriate nerve diagrams commute, the q_j respect the symmetric forms ψ_\bullet and μ_\bullet and the compatibility condition

$$q_i \circ \alpha = \beta \circ p_i$$

holds.

Let $\Psi_n(x_\bullet, \psi_\bullet) \rightarrow \Psi_n(y_\bullet, \mu_\bullet)$ be a morphism determined by the collection (p_\bullet, q_\bullet) . Since α and β are the identity maps, the compatibility condition gives $q_i = p_i$. Any morphism $\Psi_n(x_\bullet, \psi_\bullet) \rightarrow \Psi_n(y_\bullet, \mu_\bullet)$ is therefore the image of a unique isometry

$(x_\bullet, \psi_\bullet) \rightarrow (y_\bullet, \mu_\bullet)$. Here we have used that for an isometry the map $x_j \rightarrow y_j$ uniquely determines the map $x_{j'} \rightarrow y_{j'}$. It follows that Ψ_n is fully faithful.

Suppose now that we are given an object of the form (13). Consider the object of $\mathcal{U}_n(\mathcal{C})$ with underlying diagram the top row of

$$\begin{array}{ccccccc} x_0 \rightarrow \cdots \rightarrow x_{i-1} \rightarrow x'_i \rightarrow \cdots \rightarrow x'_n \rightarrow x'_{n'} \rightarrow \cdots \rightarrow x'_{i'} \rightarrow P(x_{i-1}) \rightarrow \cdots \rightarrow P(x_0) \\ \downarrow \nearrow \alpha & & \psi_{i'} \downarrow \nearrow & & & & \\ & x_i & & P(x'_{i'}) & & & \end{array}$$

and with symmetric form $(\{\Theta_\bullet\}_{[0,i]}, \{\psi_\bullet\}_{[i,i']}, \{\mathbf{1}_\bullet\}_{[i',0']})$, the subscripts indicating the index intervals to which each map applies. In this diagram the triangles define the corresponding horizontal morphisms and the unlabelled maps are the canonical ones. This object is sent by Ψ_n to

$$(x_0 \rightarrow \cdots \rightarrow x_{i-1} \rightarrow x'_i, (x'_i \rightarrow \cdots \rightarrow x'_n \rightarrow x'_{n'} \rightarrow \cdots \rightarrow x'_{i'}, \psi_\bullet); \mathbf{1}).$$

which is isomorphic via $((\mathbf{1}, \dots, \mathbf{1}, \alpha^{-1}), (\mathbf{1}, \dots, \mathbf{1}))$ to the object (13). \square

At the level of simplicial sets, the constructions of this section define a morphism $U_\bullet(\mathcal{C}) = N_\bullet^e(\mathcal{C})_h \rightarrow N_\bullet(\mathcal{C})$ which, however, is neither left nor right relative 1-Segal.

3.3. Unoriented categorified twisted cyclic nerves. Let X be a topological space with a self-homeomorphism $T : X \rightarrow X$. The T -twisted loop space of X is

$$L^T X = \{\gamma \in C^0(\mathbb{R}, X) \mid \gamma(t+1) = T(\gamma(t))\}.$$

The case of the identity map $T = \mathbf{1}_X$ recovers the ordinary loop space LX . Suppose that we are also given a continuous involution $\tau : X \rightarrow X$ which satisfies

$$\tau = T \circ \tau \circ T.$$

This, together with the orientation reversing involution $\sigma : \mathbb{R} \rightarrow \mathbb{R}, t \mapsto 1 - t$, induces an involution

$$\tau_L : L^T X \rightarrow L^T X, \quad \gamma \mapsto \tau \circ \gamma \circ \sigma^{-1},$$

the fixed point set of which is naturally interpreted as the unoriented loop space of the stack $[X/\langle T \rangle]$. The goal of this section is to construct a relative 2-Segal simplicial space which is a categorical analogue of the unoriented loop space.

Let \mathcal{C} be a small category with an endofunctor $T : \mathcal{C} \rightarrow \mathcal{C}$. The categorified T -twisted cyclic nerve of \mathcal{C} is the simplicial groupoid $\mathcal{NC}_\bullet^T(\mathcal{C})$ which assigns to $[n] \in \Delta$ the groupoid of all diagrams of the form

$$x_\bullet = \{x_0 \xrightarrow{f_0} x_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-2}} x_{n-1} \xrightarrow{f_{n-1}} x_n \xrightarrow{f_n} T(x_0)\}. \quad (14)$$

A morphism $x_\bullet \rightarrow y_\bullet$ in $\mathcal{NC}_\bullet^T(\mathcal{C})$ is a collection of isomorphisms $x_i \rightarrow y_i$, $0 \leq i \leq n$, which make the obvious diagrams commute. For $i = 1, \dots, n$ the face map $\partial_i : \mathcal{NC}_n^T(\mathcal{C}) \rightarrow \mathcal{NC}_{n-1}^T(\mathcal{C})$ omits x_i and composes f_i and f_{i-1} . The face map ∂_0 sends (14) to

$$x_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-2}} x_{n-1} \xrightarrow{f_{n-1}} x_n \xrightarrow{T(f_0) \circ f_n} T(x_1).$$

The degeneracy maps insert identity morphisms at appropriate spots.

It is proved in [5, Theorem 3.2.3] that the (non-categorified) T -twisted cyclic nerve $NC_\bullet^T(\mathcal{C})$ is a unital 2-Segal simplicial set. The analogous results holds in the categorified setting.

Theorem 3.3. *For any small category \mathcal{C} and endofunctor $T : \mathcal{C} \rightarrow \mathcal{C}$, the simplicial space $BNC_\bullet^T(\mathcal{C})$ is unital 2-Segal.*

Proof. Omitted. A similar result will be proved in Theorem 3.5 below. \square

In addition to the pair (\mathcal{C}, T) , suppose now that we are also given a duality structure (P, Θ) on \mathcal{C} and a natural transformation

$$\lambda : P \rightarrow T \circ P \circ T$$

which satisfies the compatibility condition

$$TP(\lambda_x) \circ \lambda_{PT(x)} \circ \Theta_{T(x)} = T(\Theta_x), \quad x \in \mathcal{C}. \quad (15)$$

For example, the pair $(T, \lambda) = (\mathbf{1}_{\mathcal{C}}, \text{id})$ satisfies this condition.

For each $n \geq 0$ define a contravariant functor $P_n : \mathcal{NC}_n^T(\mathcal{C}) \rightarrow \mathcal{NC}_n^T(\mathcal{C})$ by sending the diagram (14) to

$$P(x_n) \xrightarrow{P(f_{n-1})} P(x_{n-1}) \xrightarrow{P(f_{n-2})} \dots \xrightarrow{P(f_1)} P(x_1) \xrightarrow{P(f_0)} P(x_0) \xrightarrow{P_*(f_n)} TP(x_n).$$

Here $P_*(f_n)$ is the composition

$$P(x_0) \xrightarrow{\lambda_{x_0}} TPT(x_0) \xrightarrow{TP(f_n)} TP(x_n).$$

The action of P_n on morphisms is the obvious one.

Lemma 3.4. *The tuple*

$$\Theta_{n, x_\bullet} = (\Theta_{x_0}, \dots, \Theta_{x_n}, T(\Theta_{x_0})), \quad x_\bullet \in \mathcal{NC}_n^T(\mathcal{C})$$

defines a natural isomorphism $\Theta_n : \mathbf{1}_{\mathcal{NC}_n^T(\mathcal{C})} \rightarrow P_n \circ P_n$ which gives a triple $(\mathcal{NC}_\bullet^T(\mathcal{C}), P_\bullet, \Theta_\bullet)$ satisfying the hypothesis of Lemma 3.1.

Proof. We will prove that Θ_{n, x_\bullet} defines a morphism $x_\bullet \rightarrow P_n^2(x_\bullet)$; the remaining statements of the lemma can be verified directly. Keeping the notation (14), we need to show that the diagram

$$\begin{array}{ccc} x_n & \xrightarrow{f_n} & T(x_0) \\ \Theta_{x_n} \downarrow & & \downarrow T(\Theta_{x_0}) \\ P^2(x_n) & \xrightarrow{P_*^2(f_n)} & TP^2(x_0) \end{array}$$

commutes. By definition we have

$$P_*^2(f_n) = TP(\lambda_{x_0}) \circ TPTP(f_n) \circ \lambda_{P(x_n)}.$$

The natural transformation λ associates to $x_n \xrightarrow{f_n} T(x_0)$ the commutative diagram

$$\begin{array}{ccc} TPTP(x_n) & \xrightarrow{TPTP(f_n)} & TPTPT(x_0) \\ \lambda_{P(x_n)} \uparrow & & \uparrow \lambda_{PT(x_0)} \\ P^2(x_n) & \xrightarrow{P^2(f_n)} & P^2T(x_0) \end{array}$$

Combining this with the equality $P^2(f_n) \circ \Theta_{x_n} = \Theta_{T(x_0)} \circ f_n$, we compute

$$\begin{aligned} P_*^2(f_0) \circ \Theta_{x_n} &= TP(\lambda_{x_0}) \circ TPTP(f_n) \circ \lambda_{P(x_n)} \circ \Theta_{x_n} \\ &= TP(\lambda_{x_0}) \circ \lambda_{PT(x_0)} \circ \Theta_{T(x_0)} \circ f_n \\ &= T(\Theta_{x_0}) \circ f_n, \end{aligned}$$

where in the final step the compatibility condition (15) was used. \square

It follows from Lemmas 3.1 and 3.4 that $\mathcal{N}C_{\bullet}^{T,e}(\mathcal{C})$ is a simplicial groupoid with duality. Let $\mathcal{N}U_{\bullet}^T(\mathcal{C})$ be the Hermitian groupoid $\mathcal{N}C_{\bullet}^{T,e}(\mathcal{C})_h$. Using notation similar to that of Section 3.2, we write $(x_{\bullet}, \psi_{\bullet}) \in \mathcal{N}U_n^T(\mathcal{C})$ for the diagram

$$x_0 \xrightarrow{f_0} x_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} x_n \xrightarrow{f_n} x_{n'} \xrightarrow{f_{n'}} x_{(n-1)'} \xrightarrow{f_{(n-1)'}} \cdots \xrightarrow{f_{1'}} x_{0'} \xrightarrow{f_{0'}} T(x_0)$$

together with symmetric isomorphisms ψ_{\bullet} which satisfy

$$\psi_{i+1} \circ f_i = P(f_{(i+1)'}) \circ \psi_i, \quad i = 0, \dots, n-1$$

and

$$\psi_{n'} \circ f_n = P(f_n) \circ \psi_n, \quad T(\psi_0) \circ f_{0'} = P_*(f_{0'}) \circ \psi_{0'}.$$

The following statement gives a relative 2-Segal space which plays the role of the unoriented loop space.

Theorem 3.5. *For any small category with duality (\mathcal{C}, P, Θ) and endofunctor $T : \mathcal{C} \rightarrow \mathcal{C}$ with compatibility data λ as above, the morphism $BF_{\bullet} : B\mathcal{N}U_{\bullet}^T(\mathcal{C}) \rightarrow B\mathcal{N}C_{\bullet}^T(\mathcal{C})$ is a unital relative 2-Segal simplicial space.*

Proof. We need to prove that the 1-Segal morphisms

$$\Xi_n : \mathcal{N}U_n^T(\mathcal{C}) \rightarrow \mathcal{N}U_i^T(\mathcal{C}) \times_{\mathcal{N}U_{\{i\}}^T(\mathcal{C})}^{(2)} \mathcal{N}U_{n-i}^T(\mathcal{C})$$

and the relative 2-Segal morphisms

$$\Psi_n : \mathcal{N}U_n^T(\mathcal{C}) \rightarrow \mathcal{N}C_n^T(\mathcal{C}) \times_{\mathcal{N}C_{\{0,n\}}^T(\mathcal{C})}^{(2)} \mathcal{N}U_{\{0,n\}}^T(\mathcal{C})$$

are equivalences. Since the proofs are similar we will only prove the latter. Explicitly, Ψ_n sends $(x_{\bullet}, \psi_{\bullet}) \in \mathcal{N}U_n^T(\mathcal{C})$ to

$$(x_0 \rightarrow \cdots \rightarrow x_n \rightarrow T(x_0), (x_0 \rightarrow x_n \rightarrow x_{n'} \rightarrow x_{0'} \rightarrow T(x_0), \psi_{\bullet}); \mathbf{1}).$$

To see that Ψ_n is fully faithful, let $\Psi_n(x_{\bullet}, \psi_{\bullet}) \rightarrow \Psi_n(y_{\bullet}, \mu_{\bullet})$ be a morphism determined by maps $((p_0, \dots, p_n), (q_0, q_n))$, the notation an obvious modification of that from the proof of Proposition 3.2. Since the isomorphisms α_0, α_n and β_0, β_n used to define arbitrary morphisms in $\mathcal{N}C_n^T(\mathcal{C}) \times_{\mathcal{N}C_{\{0,n\}}^T(\mathcal{C})}^{(2)} \mathcal{N}U_{\{0,n\}}^T(\mathcal{C})$ are the identities in this case, the compatibility conditions imply $p_0 = q_0$ and $p_n = q_n$. Using the symmetry conditions imposed by the isometry condition, any morphism $\Psi_n(x_{\bullet}, \psi_{\bullet}) \rightarrow \Psi_n(y_{\bullet}, \mu_{\bullet})$ is therefore the image of a unique morphism $(x_{\bullet}, \psi_{\bullet}) \rightarrow (y_{\bullet}, \mu_{\bullet})$.

To prove that Ψ_n is essentially surjective, let

$$(x_0 \rightarrow \cdots \rightarrow x_n \rightarrow T(x_0), (x'_0 \rightarrow x'_n \rightarrow x'_{n'} \rightarrow x'_{0'} \rightarrow T(x'_0), \psi_{\bullet}); (\alpha_0, \alpha_n)) \quad (16)$$

be an object of $\mathcal{N}C_n^T(\mathcal{C}) \times_{\mathcal{N}C_{\{0,n\}}^T(\mathcal{C})}^{(2)} \mathcal{N}U_{\{0,n\}}^T(\mathcal{C})$. Define an object of $\mathcal{N}U_n^T(\mathcal{C})$ as the top row of the diagram

$$\begin{array}{ccccccc} x'_0 \rightarrow x_1 \rightarrow \cdots \rightarrow x_{n-1} \rightarrow x'_n \rightarrow x'_{n'} \rightarrow P(x_{n-1}) \rightarrow \cdots \rightarrow P(x_1) \rightarrow x'_{0'} \rightarrow T(x'_0) \\ \alpha_0^{-1} \downarrow \nearrow & & \searrow \nearrow & \alpha_n \downarrow \nearrow & \psi_{n'} \downarrow \nearrow & & \downarrow \nearrow \psi_{0'}^{-1} \\ x_0 & & x_n & P(x'_n) & & & P(x'_0) \end{array}$$

with symmetric form $(\{\Theta_{\bullet}\}_{[1,n-1]}, \{\psi_{\bullet}\}_{\{0,n,n',0'\}}, \{\mathbf{1}_{\bullet}\}_{[(n-1)',1']})$. The image of this object under Ψ_n is

$$(x'_0 \rightarrow x_1 \rightarrow \cdots \rightarrow x_{n-1} \rightarrow x'_n \rightarrow T(x'_0), (x'_0 \rightarrow x'_n \rightarrow x'_{n'} \rightarrow x'_{0'} \rightarrow T(x'_0), \psi_{\bullet}); (\mathbf{1}, \mathbf{1})).$$

Then the tuple $(\alpha_0^{-1}, \mathbf{1}, \dots, \mathbf{1}, \alpha_n^{-1}, (\mathbf{1}, \dots, \mathbf{1}))$ defines an isomorphism from the previous object to the object (16). Hence Ψ_n is an equivalence. \square

Remark. The homotopical counterpart of the $\mathrm{SO}(2)$ -action on LX is the cyclic structure of $NC_\bullet(\mathcal{C})$. The $\mathrm{SO}(2)$ -action extends to an $\mathrm{O}(2)$ -action by allowing orientation reversal of loops and corresponds to the dihedral structure of $NC_\bullet(\mathcal{C})$ induced by a (strict) duality P on \mathcal{C} . More generally, $NC_\bullet^T(\mathcal{C})$ has a paradihedral or, if $T^r = 1$ for some $r \geq 2$, an r -dihedral structure. Similar comments apply to categorified nerves. See [7, §I.6] for further discussion.

3.4. The \mathcal{R}_\bullet -construction. The goal of this section is to prove that the \mathcal{R}_\bullet -construction (also called the Hermitian \mathcal{S}_\bullet -construction) from Grothendieck-Witt theory is relative 2-Segal over the Waldhausen \mathcal{S}_\bullet -construction. We work in the proto-exact setting of Section 1.4.

Definition. A proto-exact category with duality is a category with duality (\mathcal{C}, P, Θ) such that \mathcal{C} is proto-exact and P satisfies the following properties:

- (i) $P(0) \simeq 0$,
- (ii) a morphism $U \xrightarrow{\phi} V$ is in \mathfrak{I} if and only if $P(V) \xrightarrow{P(\phi)} P(U)$ is in \mathfrak{D} , and
- (iii) P sends biCartesian squares to biCartesian squares.

Let N be a symmetric form in a proto-exact category with duality \mathcal{C} . The orthogonal U^\perp of an inflation $i : U \rightarrowtail N$ is defined as the pullback

$$\begin{array}{ccc} U^\perp & \rightarrowtail & N \\ \downarrow & & \downarrow \\ 0 & \rightarrowtail & P(U) \end{array}$$

The inflation i is called isotropic if the composition $P(i)\psi_N i$ is zero and the canonical monomorphism $U \rightarrow U^\perp$ is an inflation.

In this section we will make the following assumption on \mathcal{C} .

Reduction Assumption. Let N be a symmetric form in \mathcal{C} and let $U \rightarrowtail N$ be isotropic with orthogonal $k : U^\perp \rightarrowtail N$. Define $M \in \mathcal{C}$ as the pushout

$$\begin{array}{ccc} U & \rightarrowtail & U^\perp \\ \downarrow & & \downarrow \pi \\ 0 & \rightarrowtail & M \end{array}$$

Then there is a unique symmetric form ψ_M on M such that $P(k)\psi_N k = P(\pi)\psi_M \pi$.

The symmetric form (M, ψ_M) is called the isotropic reduction of N by U and is denoted by $N//U$. When \mathcal{C} is exact the Reduction Assumption is known to hold; see [27, Lemma 5.2], [32, Lemma 2.6]. In a number of (non-exact) proto-exact examples, such as $\mathbf{Rep}_{\mathbb{F}_1}(Q)$, the Reduction Assumption also holds, as can be verified directly.

The category $[n]$ has a strict duality structure given on objects by $i \mapsto n - i$. The category $[\mathbf{Ar}_n, \mathcal{C}]$ and its full subcategory $\mathcal{W}_n(\mathcal{C})$ therefore inherit duality structures. Moreover, the duality structures on $\mathcal{W}_\bullet(\mathcal{C})$ satisfy the assumptions of Lemma 3.1 so that $\mathcal{W}_\bullet^e(\mathcal{C})$ is a simplicial object of \mathbf{CatD} . Explicitly, the dual of a diagram $\{A_{\{p,q\}}\}_{0 \leq p \leq q \leq 0'} \in \mathcal{W}_n^e(\mathcal{C})$ is the diagram $\{P(A_{\{q',p'\}})\}_{0 \leq p \leq q \leq 0'}$ and the double dual identification at (p, q) is $\Theta_{A_{\{p,q\}}}$.

The \mathcal{R}_\bullet -construction [36], [12], denoted by $\mathcal{R}_\bullet(\mathcal{C})$, is the Hermitian groupoid $\mathcal{W}_\bullet^e(\mathcal{C})_h$.¹ Objects of $\mathcal{R}_n(\mathcal{C})$ are diagrams $\{A_{\{p,q\}}\}_{0 \leq p \leq q \leq 0'} \in \mathcal{W}_n^e(\mathcal{C})$ together with

¹In [12] the notation $\mathcal{R}_\bullet(\mathcal{C})$ is used for $\mathcal{W}_\bullet^e(\mathcal{C})$, whereas what we call $\mathcal{R}_\bullet(\mathcal{C})$ is denoted by $\mathcal{R}_\bullet^h(\mathcal{C})$.

symmetric isomorphisms $\psi_{p,q} : A_{\{p,q\}} \rightarrow P(A_{\{q',p'\}})$ which make all appropriate diagrams commute. In particular,

- (i) for every $1 \leq i \leq n$ the pair $(A_{\{i,i'\}}, \psi_{i,i'})$ is a symmetric form,
- (ii) for every $0 \leq i \leq j \leq n$ the inflation $A_{\{i,j\}} \twoheadrightarrow A_{\{i,i'\}}$ is isotropic with orthogonal $A_{\{i,j'\}} \twoheadrightarrow A_{\{i,i'\}}$, and
- (iii) for every $0 \leq i \leq j \leq n$ the symmetric form $(A_{\{j,j'\}}, \psi_{j,j'})$ is isometric to the reduction $(A_{\{i,i'\}}, \psi_{i,i'}) // A_{\{i,j\}}$.

For example, an object of $\mathcal{R}_0(\mathcal{C})$ is a diagram

$$\begin{array}{c} 0 \twoheadrightarrow A_{\{0,0'\}} \\ \downarrow \\ 0 \end{array}$$

together with a symmetric form on $A_{\{0,0'\}}$. Similarly, an object of $\mathcal{R}_1(\mathcal{C})$ is a diagram

$$\begin{array}{ccccccc} 0 & \twoheadrightarrow & A_{\{0,1\}} & \twoheadrightarrow & A_{\{0,1'\}} & \twoheadrightarrow & A_{\{0,0'\}} \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & \twoheadrightarrow & A_{\{1,1'\}} & \twoheadrightarrow & A_{\{1,0'\}} \\ & & & & \downarrow & & \downarrow \\ & & & & 0 & \twoheadrightarrow & A_{\{1',0'\}} \\ & & & & & & \downarrow \\ & & & & & & 0 \end{array}$$

all of whose squares are biCartesian, together with the data of

- (i) a symmetric form on $A_{\{0,0'\}}$ such that $A_{\{0,1\}} \twoheadrightarrow A_{\{0,0'\}}$ is isotropic with orthogonal $A_{\{0,1'\}} \twoheadrightarrow A_{\{0,0'\}}$,
- (ii) a symmetric form on $A_{\{1,1'\}}$ presenting $A_{\{1,1'\}}$ as $A_{\{0,0'\}} // A_{\{0,1\}}$, and
- (iii) isomorphisms $A_{\{0,1\}} \simeq P(A_{\{1',0'\}})$ and $A_{\{0,1'\}} \simeq P(A_{\{1,0'\}})$ such that each morphism above the diagonal agrees with the corresponding morphism below the diagonal.

The forgetful map

$$F_\bullet : \mathcal{R}_\bullet(\mathcal{C}) \rightarrow \mathcal{S}_\bullet(\mathcal{C}), \quad \mathcal{R}_n(\mathcal{C}) \ni \{A_{\{p,q\}}, \psi_{p,q}\}_{0 \leq p \leq q \leq 0'} \mapsto \{A_{\{p,q\}}\}_{0 \leq p \leq q \leq n}$$

is a morphism of simplicial groupoids.

Theorem 3.6. *Let \mathcal{C} be a proto-exact category with duality which satisfies the Reduction Assumption. Then the morphism $BF_\bullet : BR_\bullet(\mathcal{C}) \rightarrow BS_\bullet(\mathcal{C})$ is a unital relative 2-Segal simplicial space.*

Proof. For each $n \geq 0$ let $\mathcal{I}_n(\mathcal{C})$ be the groupoid of isotropic n -flags in \mathcal{C} . An object of $\mathcal{I}_n(\mathcal{C})$ is a diagram

$$0 \twoheadrightarrow A_1 \twoheadrightarrow \cdots \twoheadrightarrow A_n \twoheadrightarrow A_{n'} \twoheadrightarrow \cdots \twoheadrightarrow A_{1'} \twoheadrightarrow (A_{0'}, \psi_{0'}) \quad (17)$$

with $(A_{0'}, \psi_{0'})$ a symmetric form such that, for each $0 \leq i \leq n$, the inflation $A_i \twoheadrightarrow A_{0'}$ is isotropic with orthogonal $A_{i'} \twoheadrightarrow A_{0'}$. We claim that the forgetful functor

$$\nu_n : \mathcal{R}_n(\mathcal{C}) \rightarrow \mathcal{I}_n(\mathcal{C}), \quad \{A_{\{i,j\}}, \psi_{i,j}\}_{0 \leq i \leq j \leq 0'} \mapsto \{A_{\{0,j\}}, \psi_{0,j}\}_{0 \leq j \leq 0'}$$

is an equivalence. We construct a quasi-inverse η_n of ν_n as follows. Given an isotropic flag (17), put $A_{\{0,i\}} = A_k$, $1 \leq k \leq 1'$, and $(A_{\{0,0'\}}, \psi_{0,0'}) = (A_{0'}, \psi_{0'})$. For each $1 \leq k \leq 1'$ let $A_{\{1,k\}}$ be the pushout

$$\begin{array}{ccc} A_{\{0,1\}} & \xrightarrow{\quad} & A_{\{0,k\}} \\ \downarrow & & \downarrow \\ 0 & \xrightarrow{\quad\quad\quad} & A_{\{1,k\}} \end{array}$$

By the Reduction Assumption the symmetric form $(A_{\{0,0'\}}, \psi_{0,0'})$ induces a unique compatible symmetric form on $A_{\{1,1'\}}$. Define also $A_{\{1,0'\}} = P(A_{\{0,1'\}})$ and let $A_{\{0,0'\}} \twoheadrightarrow A_{\{1,0'\}}$ be the composition

$$A_{\{0,0'\}} \xrightarrow{\psi_{0,0'}} P(A_{\{0,0'\}}) \twoheadrightarrow P(A_{\{0,1'\}}).$$

This construction defines the top two rows of $\eta_n(A_{\{0,\bullet\}})$ and can be iterated to define the remaining $n - 2$ rows. It is clear that ν_n is a quasi-inverse to η_n .

We can now prove the theorem. The 1-Segal morphism for $\mathcal{R}_\bullet(\mathcal{C})$ is

$$\Xi_n : \mathcal{R}_n(\mathcal{C}) \rightarrow \mathcal{R}_{\{0,\dots,i\}}(\mathcal{C}) \times_{\mathcal{R}_{\{i\}}(\mathcal{C})}^{(2)} \mathcal{R}_{\{i,\dots,n\}}(\mathcal{C}).$$

Arguing as in the proof of Theorem 2.7 and using that ν_n is an equivalence, to prove that Ξ_n is an equivalence it suffices to prove that the functor

$$\tilde{\Xi}_n : \mathcal{I}_n(\mathcal{C}) \rightarrow \mathcal{I}_i(\mathcal{C}) \times_{\mathcal{I}_{\{i\}}(\mathcal{C})}^{(2)} \mathcal{I}_{n-i}(\mathcal{C})$$

is an equivalence. A quasi-inverse of $\tilde{\Xi}_n$ is defined by assigning to a pair

$$(0 \twoheadrightarrow A'_1 \twoheadrightarrow \dots \twoheadrightarrow A'_i \twoheadrightarrow A'_{i'} \twoheadrightarrow \dots \twoheadrightarrow A'_{1'} \twoheadrightarrow (A'_{0'}, \psi'_{0'})) \in \mathcal{I}_i(\mathcal{C})$$

and

$$(0 \twoheadrightarrow A''_{i+1} \twoheadrightarrow \dots \twoheadrightarrow A''_n \twoheadrightarrow A''_{n'} \twoheadrightarrow \dots \twoheadrightarrow A''_{(i+1)''} \twoheadrightarrow (A''_{i'}, \psi''_{i'})) \in \mathcal{I}_{n-i}(\mathcal{C})$$

together with an isometry $(A'_{0'}, \psi'_{0'}) // A'_i \simeq (A''_{i'}, \psi''_{i'})$ the object

$$(0 \twoheadrightarrow A'_1 \twoheadrightarrow \dots \twoheadrightarrow A'_n \twoheadrightarrow A'_{n'} \twoheadrightarrow \dots \twoheadrightarrow A'_{1'} \twoheadrightarrow (A'_{0'}, \psi'_{0'})) \in \mathcal{I}_n(\mathcal{C})$$

where A'_j , $i + 1 \leq j \leq (i + 1)'$, is defined to be the pullback

$$\begin{array}{ccc} A'_j & \xrightarrow{\quad\quad} & A'_{i'} \\ \downarrow & & \downarrow \\ A''_j & \xrightarrow{\quad\quad} & A''_{i'} \end{array}$$

That this is indeed a quasi-inverse follows from the Hermitian variant of the Second Isomorphism Theorem [27, Proposition 6.5], which generalizes to the proto-exact setting under the Reduction Assumption.

Arguing in the same way, to prove that the relative 2-Segal map

$$\Psi_n : \mathcal{R}_n(\mathcal{C}) \rightarrow \mathcal{S}_n(\mathcal{C}) \times_{\mathcal{S}_{\{0,n\}}(\mathcal{C})}^{(2)} \mathcal{R}_{\{0,n\}}(\mathcal{C})$$

is an equivalence it suffices to prove that the functor

$$\tilde{\Psi}_n : \mathcal{I}_n(\mathcal{C}) \rightarrow \mathcal{F}_n(\mathcal{C}) \times_{\mathcal{F}_{\{0,n\}}(\mathcal{C})}^{(2)} \mathcal{I}_{\{0,n\}}(\mathcal{C})$$

is an equivalence, which is obvious.

Finally, relative unitality is the condition that, for each $0 \leq i \leq n - 1$, the map

$$\Upsilon_n : \mathcal{R}_{n-1}(\mathcal{C}) \rightarrow \mathcal{S}_{\{i\}}(\mathcal{C}) \times_{\mathcal{S}_{\{i,i+1\}}(\mathcal{C})}^{(2)} \mathcal{R}_n(\mathcal{C})$$

is an equivalence. Note that $\mathcal{S}_0(\mathcal{C})$ is a point, $\mathcal{S}_1(\mathcal{C})$ is the maximal groupoid of \mathcal{C} and the map $\mathcal{S}_0(\mathcal{C}) \xrightarrow{s_0} \mathcal{S}_1(\mathcal{C})$ sends the point to the zero object. An object of $\mathcal{S}_{\{i\}}(\mathcal{C}) \times_{\mathcal{S}_{\{i,i+1\}}(\mathcal{C})}^{(2)} \mathcal{R}_n(\mathcal{C})$ is therefore an object of $\mathcal{R}_n(\mathcal{C})$ whose maps between the i th and $(i+1)$ st rows/columns are the identities. It follows easily from this that Υ_n is an equivalence. \square

Remark. For comparison, the Waldhausen space $BS_{\bullet}(\mathcal{C})$ of an exact category is 1-Segal if and only if all short exact sequence in \mathcal{C} are split.

When \mathcal{C} is exact, the morphism $F_{\bullet} : \mathcal{R}_{\bullet}(\mathcal{C}) \rightarrow \mathcal{S}_{\bullet}(\mathcal{C})$ is closely related to the Grothendieck-Witt theory of \mathcal{C} [33]. Precisely, the Grothendieck-Witt space $GW(\mathcal{C})$ is the homotopy fibre over 0 of the map $|BF_{\bullet}| : |BR_{\bullet}(\mathcal{C})| \rightarrow |BS_{\bullet}(\mathcal{C})|$. Then the higher Grothendieck-Witt groups of \mathcal{C} are given by $GW_i(\mathcal{C}) = \pi_i GW(\mathcal{C})$, $i \geq 0$.

4. APPLICATIONS

4.1. Categorical Hall algebra representations. After reviewing the relationship between 2-Segal spaces and categorical Hall algebras [5], in this section we construct from a relative 2-Segal space a categorical representation of the Hall algebra of the base.

Fix a combinatorial model category \mathcal{C} , such as \mathbb{S} with its Kan model structure or \mathbf{Grpd} with its Bousfield model structure; the latter will be the main source of examples. While the Quillen model structure on \mathbf{Top} is not combinatorial, it is Quillen equivalent to \mathbb{S} so that the results of this section, for all intents and purposes, also apply to simplicial spaces. Write $\text{pt} \in \mathcal{C}$ for the final object.

Let $\mathbf{Span}(\mathcal{C})$ be the bicategory of spans in \mathcal{C} . Objects of $\mathbf{Span}(\mathcal{C})$ are simply objects of \mathcal{C} while 1-morphisms of $\mathbf{Span}(\mathcal{C})$ are spans, that is, diagrams in \mathcal{C} of the form

$$A \xleftarrow{l} X \xrightarrow{r} B.$$

Composition of spans is given by homotopy pullback. A 2-morphism in $\mathbf{Span}(\mathcal{C})$ is a homotopy commutative diagram

$$\begin{array}{ccccc} & & X & & \\ & \swarrow l & \downarrow & \searrow r & \\ A & & & & B \\ & \swarrow l' & \downarrow & \searrow r' & \\ & & X' & & \end{array}$$

in \mathcal{C} . Give $\mathbf{Span}(\mathcal{C})$ the Cartesian monoidal structure.

Associated to the bicategory $\mathbf{Span}(\mathcal{C})$ is the category $\mathbf{Span}(\mathcal{C})^{\sim}$ having the same objects as $\mathbf{Span}(\mathcal{C})$ and with morphisms the 2-isomorphism classes spans. The constructions of this section are considerably simplified if one uses $\mathbf{Span}(\mathcal{C})^{\sim}$ in place of $\mathbf{Span}(\mathcal{C})$, the downside being that less of the 2-Segal structure is used.

Definition ([5, §8.1]). *A transfer structure on \mathcal{C} is a pair $(\mathcal{S}, \mathcal{P})$ consisting of collections of morphisms \mathcal{S} and \mathcal{P} in \mathcal{C} called smooth and proper, respectively, which satisfy the following properties:*

- (1) *Both collections \mathcal{S} and \mathcal{P} are closed under composition.*

(2) Given a homotopy Cartesian diagram

$$\begin{array}{ccc} X & \xrightarrow{s} & Y \\ p \downarrow & & \downarrow p' \\ X' & \xrightarrow{s'} & Y' \end{array} \quad (18)$$

in \mathcal{C} with $s' \in \mathcal{S}$ and $p' \in \mathcal{P}$, then $s \in \mathcal{S}$ and $p \in \mathcal{P}$.

Example. The pair $(\mathcal{S}, \mathcal{P}) = (\mathcal{C}, \mathcal{C})$ is called the trivial transfer structure on \mathcal{C} . \triangleleft

Spans of the form $A \xleftarrow{l} X \xrightarrow{r} B$ with $l \in \mathcal{S}$ and $r \in \mathcal{P}$ are called $(\mathcal{S}, \mathcal{P})$ -admissible and form a subbcategory $\text{Span}(\mathcal{S}, \mathcal{P}) \subset \text{Span}(\mathcal{C})$.

Theorem 4.1 ([5, Proposition 8.1.7]). *Let X_\bullet be a 2-Segal object of \mathcal{C} for which the span*

$$m = \left\{ X_{\{0,1\}} \xleftarrow{(\partial_2, \partial_0)} X_{\{0,1,2\}} \xrightarrow{\partial_1} X_{\{0,2\}} \right\} \in \text{Hom}_{\text{Span}(\mathcal{C})}(X_1 \times X_1, X_1) \quad (19)$$

is $(\mathcal{S}, \mathcal{P})$ -admissible. Then (X_1, m) is a semigroup in $\text{Span}(\mathcal{S}, \mathcal{P})$. Moreover, if X_\bullet is unital and the span

$$I_X = \left\{ \text{pt} \xleftarrow{\text{can}} X_0 \xrightarrow{s_0} X_1 \right\} \in \text{Hom}_{\text{Span}(\mathcal{C})}(\text{pt}, X_1)$$

is $(\mathcal{S}, \mathcal{P})$ -admissible, then (X_1, m, I_X) is a monoid in $\text{Span}(\mathcal{S}, \mathcal{P})$.

We call $\mathcal{H}(X_\bullet) = (X_1, m)$ the $(\mathcal{S}, \mathcal{P})$ -universal Hall algebra of X_\bullet . Slightly abusively, we have omitted from the notation the associativity isomorphism a for m . Note that at the universal level the transfer structure $(\mathcal{S}, \mathcal{P})$ simply determines the ambient category to which $\mathcal{H}(X_\bullet)$ belongs. A more important role is played by $(\mathcal{S}, \mathcal{P})$ when passing to concrete realizations of Hall algebras. To explain this procedure, fix a monoidal category $(\mathcal{V}, \otimes, \mathbf{1}_{\mathcal{V}})$.

Definition ([5, §8.1]). *A \mathcal{V} -valued theory with transfer on \mathcal{C} is the data of*

- (1) *a transfer structure $(\mathcal{S}, \mathcal{P})$ on \mathcal{C} ,*
- (2) *a contravariant functor $(-)^* : \mathcal{S} \rightarrow \mathcal{V}$ and a covariant functor $(-)_* : \mathcal{P} \rightarrow \mathcal{V}$ with common values on objects (denoted by \mathfrak{h}) and which map weak equivalences to isomorphisms, and*
- (3) *an isomorphism $\mathfrak{h}(\text{pt}) \simeq \mathbf{1}_{\mathcal{V}}$ and multiplicativity data for \mathfrak{h} , that is, natural maps*

$$\mathfrak{h}(X) \otimes \mathfrak{h}(Y) \rightarrow \mathfrak{h}(X \times Y)$$

which satisfy associativity and unitality conditions

such that for a homotopy Cartesian diagram (18) with $s, s' \in \mathcal{S}$ and $p, p' \in \mathcal{P}$, we have $p'_ \circ s^* = s'^* \circ p_*$.*

Applying a \mathcal{V} -valued theory with transfer \mathfrak{h} to the $(\mathcal{S}, \mathcal{P})$ -universal Hall algebra gives an algebra in \mathcal{V} , denoted by

$$\mathcal{H}(X_\bullet; \mathfrak{h}) = (\mathfrak{h}(X_1), \partial_{1*} \circ (\partial_2 \times \partial_0)^*).$$

Dually, if the opposite span $m^{\text{op}} \in \text{Hom}_{\text{Span}(\mathcal{C})}(X_1 \times X_1, X_1)$ is $(\mathcal{S}, \mathcal{P})$ -admissible, then (X_1, m^{op}) is a cosemigroup in $\text{Span}(\mathcal{S}, \mathcal{P})$, called the $(\mathcal{S}, \mathcal{P})$ -universal Hall coalgebra. Passing to theories with transfer then gives coalgebras in monoidal categories. These statements can be proved in the same way as Theorem 4.1.

Example. Let $\mathcal{S}_\bullet(\mathcal{C})$ be the Waldhausen groupoid of an essentially small exact category \mathcal{C} . Suppose that \mathcal{C} is finitary in the sense that $\oplus_{n \geq 0} \text{Ext}_{\text{env}(\mathcal{C})}^n(U, V)$ is finite for all $U, V \in \mathcal{C}$, where $\text{env}(\mathcal{C})$ denotes the abelian envelope of \mathcal{C} [28]. Then $\mathcal{H}(\mathcal{S}_\bullet(\mathcal{C}))$ categorifies the Hall algebra of \mathcal{C} , as defined in various contexts in [30], [13], [31]. To see this, let k be a field of characteristic zero. Consider the transfer structure in which \mathcal{S} and \mathcal{P} are the collections of weakly proper and locally proper morphisms, respectively (see [5, §8.2]). A Vect_k -valued theory with transfer \mathfrak{F}_0 is then defined by taking finitely supported k -valued orbifold functions. The resulting k -algebra $\mathcal{H}(\mathcal{S}_\bullet(\mathcal{C}); \mathfrak{F}_0)$ is the standard Hall algebra of \mathcal{C} . Explicitly, $\mathcal{H}(\mathcal{S}_\bullet(\mathcal{C}); \mathfrak{F}_0)$ is the k -vector space with basis $\{\mathbf{1}_U\}_{U \in \pi_0(\mathcal{S}_1(\mathcal{C}))}$ and product

$$\mathbf{1}_U \cdot \mathbf{1}_V = \sum_{W \in \pi_0(\mathcal{S}_1(\mathcal{C}))} F_{U,V}^W \mathbf{1}_W$$

where $F_{U,V}^W$ is the number of admissible subobjects of W which are isomorphic to U and have quotient isomorphic to V .

For certain categories \mathcal{C} , such as $\text{Coh}(X)$ for a smooth projective variety X or $\text{mod}(A)$ for a finitely generated algebra A , the Waldhausen \mathcal{S}_\bullet -construction defines a 2-Segal simplicial Artin stack. In this way the perverse sheaf theoretic [23], motivic [17], [20] and cohomological [21] Hall algebras of \mathcal{C} can be recovered. For details see [5, §8.5]. \triangleleft

By using relative 2-Segal spaces we can easily modify the above results to construct module objects.

Theorem 4.2. *Let X_\bullet be as in Theorem 4.1. Suppose that $F_\bullet : Y_\bullet \rightarrow X_\bullet$ is a relative 2-Segal object of \mathcal{C} . If the span*

$$\mu_l = \left\{ X_{\{0,1\}} \times Y_{\{1\}} \xleftarrow{(F_1, \partial_0)} Y_{\{0,1\}} \xrightarrow{\partial_1} Y_{\{1\}} \right\} \in \text{Hom}_{\text{Span}(\mathcal{C})}(X_1 \times Y_0, Y_0) \quad (20)$$

is $(\mathcal{S}, \mathcal{P})$ -admissible, then (Y_0, μ_l) is a left (X_1, m) -module in $\text{Span}(\mathcal{S}, \mathcal{P})$. Moreover, if F_\bullet is unital, then (Y_0, μ_l) is a left (X_1, m, I_X) -module. Analogous statements hold for the right span

$$\mu_r = \left\{ Y_{\{0\}} \times X_{\{0,1\}} \xleftarrow{\partial_1 \times F_1} Y_{\{0,1\}} \xrightarrow{\partial_0} Y_{\{1\}} \right\} \in \text{Hom}_{\text{Span}(\mathcal{C})}(Y_0 \times X_1, Y_0).$$

Proof. We will prove the theorem for the left span μ_l . It follows directly from the definitions that we have

$$\mu_l \circ (m \otimes \mathbf{1}_{Y_0}) = \left\{ \begin{array}{c} (X_{\{0,1,2\}} \times Y_{\{2\}}) \times_{X_{\{0,2\}} \times Y_{\{2\}}}^R Y_{\{0,2\}} \longrightarrow Y_{\{0,2\}} \longrightarrow Y_{\{0\}} \\ \downarrow \\ X_{\{0,1,2\}} \times Y_{\{2\}} \longrightarrow X_{\{0,2\}} \times Y_{\{2\}} \\ \downarrow \\ X_{\{0,1\}} \times X_{\{1,2\}} \times Y_{\{2\}} \end{array} \right\}$$

and

$$\mu_l \circ (\mathbf{1}_{X_1} \otimes \mu_l) = \left\{ \begin{array}{c} (X_{\{0,1\}} \times Y_{\{1,2\}}) \times_{X_{\{0,1\}} \times Y_{\{1\}}}^R Y_{\{0,1\}} \longrightarrow Y_{\{0,1\}} \longrightarrow Y_{\{0\}} \\ \downarrow \\ X_{\{0,1\}} \times Y_{\{1,2\}} \longrightarrow X_{\{0,1\}} \times Y_{\{1\}} \\ \downarrow \\ X_{\{0,1\}} \times X_{\{1,2\}} \times Y_{\{2\}} \end{array} \right\}$$

as spans $X_1 \times X_1 \times Y_0 \rightarrow Y_0$. We claim that both $\mu_l \circ (m \otimes \mathbf{1}_{Y_0})$ and $\mu_l \circ (\mathbf{1}_{X_1} \otimes \mu_l)$ are 2-isomorphic to the span

$$\sigma_l = \left\{ X_{\{0,1\}} \times X_{\{1,2\}} \times Y_{\{2\}} \xleftarrow{(F_{\{0,1\}} \circ \partial_2, F_{\{1,2\}} \circ \partial_0, \partial_0 \circ \partial_1)} Y_{\{0,1,2\}} \xrightarrow{\partial_1 \circ \partial_2} Y_{\{0\}} \right\}.$$

Indeed, the composition of functors

$$\begin{aligned} Y_{\{0,1,2\}} &\xrightarrow{\alpha_1} (X_{\{0,1,2\}} \times Y_{\{2\}}) \times_{X_{\{0,2\}} \times Y_{\{2\}}}^R Y_{\{0,2\}} \\ &\rightarrow X_{\{0,1,2\}} \times_{X_{\{0,2\}}}^R Y_{\{0,2\}} \end{aligned}$$

is a weak equivalence by the relative 2-Segal condition on F_\bullet . Since the second functor is a weak equivalence for trivial reasons, it follows that α_1 is a weak equivalence and so defines a 2-isomorphism $\sigma_l \rightarrow \mu_l \circ (m \otimes \mathbf{1}_{Y_0})$. Similarly, the composition

$$\begin{aligned} Y_{\{0,1,2\}} &\xrightarrow{\alpha_2} (X_{\{0,1\}} \times Y_{\{1,2\}}) \times_{X_{\{0,1\}} \times Y_{\{1\}}}^R Y_{\{0,1\}} \\ &\rightarrow Y_{\{1,2\}} \times_{Y_{\{1\}}}^R Y_{\{0,1\}} \end{aligned}$$

is a weak equivalence by the 1-Segal condition on Y_\bullet . It follows that α_2 defines a 2-isomorphism $\sigma_l \rightarrow \mu_l \circ (\mathbf{1}_{X_1} \otimes \mu_l)$. Consider then the composition

$$\alpha : \mu_l \circ (m \otimes \mathbf{1}_{Y_0}) \xrightarrow{\alpha_1^{-1}} \sigma_l \xrightarrow{\alpha_2} \mu_l \circ (\mathbf{1}_{X_1} \otimes \mu_l).$$

We claim that α is a module associator for the left action of (X_1, m) on Y_0 determined by μ_l . Without the unital assumption, this amounts to verifying that α satisfies module theoretic Mac Lane coherence; see [25, §2.3] or diagram (21) below for the precise condition. We will verify this using the combinatorial setting of Section 2.3. The poset of the 11 symmetric polyhedral subdivisions of the symmetric octagon P_3 , ordered by refinement, is illustrated in Figure 4. At the level of the map $F_\bullet : Y_\bullet \rightarrow X_\bullet$, each node \mathcal{P} of Figure 4 defines a span

$$\{X_1 \times_{X_0} \times_{X_1} \times_{X_0} X_1 \times_{X_0} Y_0 \leftarrow (\Delta^{\mathcal{P}}, F_\bullet)_R \rightarrow Y_0\}.$$

Similarly, each arrow of Figure 4 defines a 2-isomorphism of spans. The spans associated to the five vertices of the pentagon are precisely the spans appearing in the module theoretic Mac Lane coherence diagram and the composed 2-isomorphisms along the edges of the pentagon are the arrows in the coherence diagram. It follows that module theoretic Mac Lane coherence holds. Hence (Y_0, μ_l) is a left (X_1, m) -module.

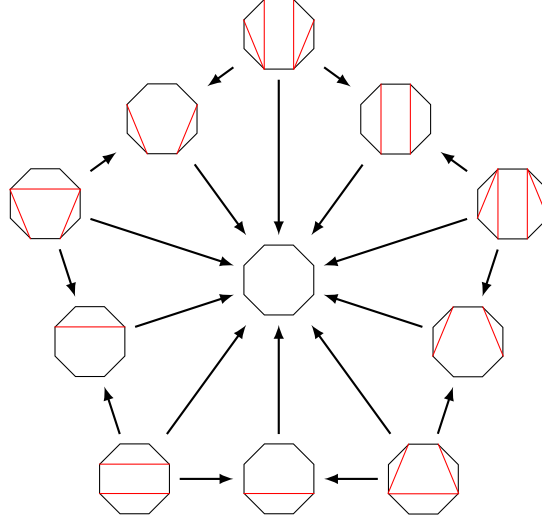
Finally, in the unital situation the action of I_X on Y_0 is given by the span

$$\begin{array}{ccccc} (X_{\{0\}} \times Y_{\{1\}}) \times_{X_{\{0,1\}} \times Y_{\{1\}}}^R Y_{\{0,1\}} & \longrightarrow & Y_{\{0,1\}} & \longrightarrow & Y_{\{0\}} \\ \downarrow & & \downarrow & & \\ X_{\{0\}} \times Y_{\{1\}} & \longrightarrow & X_{\{0,1\}} \times Y_{\{1\}} & & \\ \downarrow & & & & \\ \text{pt} \times Y_{\{1\}} & & & & \end{array}$$

which is 2-isomorphic to the identity span $Y_0 \xleftarrow{\mathbf{1}_{Y_0}} Y_0 \xrightarrow{\mathbf{1}_{Y_0}} Y_0$. Indeed, the composition of functors

$$\begin{aligned} Y_{\{0\}} &\xrightarrow{I_Y} (X_{\{0\}} \times Y_{\{1\}}) \times_{X_{\{0,1\}} \times Y_{\{1\}}}^R Y_{\{0,1\}} \\ &\rightarrow X_{\{0\}} \times_{X_{\{0,1\}}}^R Y_{\{0,1\}} \end{aligned}$$

is an equivalence by the unital relative 2-Segal condition while the second functor is trivially an equivalence. To complete the proof we need to verify that I_Y is compatible with the module associator in the sense that diagram (22) below commutes.

FIGURE 4. The poset of symmetric polyhedral subdivisions of P_3 .

This is a straightforward exercise which can be completed in much the same way as the above verification of Mac Lane coherence. \square

The left $\mathcal{H}(X_\bullet)$ -module $\mathcal{M}(Y_\bullet) = (Y_0, \mu_l)$ is called the $(\mathcal{S}, \mathcal{P})$ -universal left Hall module of F_\bullet . From a \mathcal{V} -valued theory with transfer \mathfrak{h} we obtain a left $\mathcal{H}(X_\bullet; \mathfrak{h})$ -module $\mathcal{M}(Y_\bullet; \mathfrak{h})$. In the same way we get right modules over $\mathcal{H}(X_\bullet)$ and $\mathcal{H}(X_\bullet; \mathfrak{h})$. However, it is easy to see that the left and right module structures do not define a bimodule structure on $\mathcal{M}(Y_\bullet; \mathfrak{h})$.

Example. By Proposition 2.5 the right path space $F_\bullet^\triangleright : P^\triangleright X_\bullet \rightarrow X_\bullet$ is relative 2-Segal. Since $F_n^\triangleright = \partial_{n'}$, the spans μ_l and m are equal. Hence $\mathcal{M}(X_\bullet^\triangleright) = \mathcal{H}(X_\bullet)$ as left $\mathcal{H}(X_\bullet)$ -modules. On the other hand, the right $\mathcal{H}(X_\bullet)$ -module structure on $\mathcal{M}(X_\bullet^\triangleright)$ is closely related to the coproduct Δ on $\mathcal{H}(X_\bullet)$. Consider for example $\mathcal{H}(\mathcal{S}_\bullet(\mathcal{C}); \mathfrak{F}_0)$. Recall that the Green bilinear form on $\mathcal{H}(\mathcal{S}_\bullet(\mathcal{C}); \mathfrak{F}_0)$, defined by $(\mathbf{1}_U, \mathbf{1}_V) = \delta_{U,V} \cdot |\text{Aut}(U)|^{-1}$, satisfies the Hopf property

$$(\mathbf{1}_U \otimes \mathbf{1}_V, \Delta \mathbf{1}_W) = (\mathbf{1}_U \cdot \mathbf{1}_V, \mathbf{1}_W).$$

Then the right and left $\mathcal{H}(\mathcal{S}_\bullet(\mathcal{C}); \mathfrak{F}_0)$ -actions are adjoint with respect to $(-, -)$. \triangleleft

Remark. The universal Hall algebra $\mathcal{H}(X_\bullet)$ is itself a $\mathcal{H}(X_\bullet)$ -bimodule which, however, does not arise from a relative 2-Segal space over X_\bullet .

Remark. A general description of the relationship between the right and left module structures on $\mathcal{M}(Y_\bullet)$ for a general 2-Segal space is given in [42, §5.6].

Example. Let $\mathcal{S}_\bullet^{\text{st-fr}}(\mathcal{C}_\phi^{Z\text{-ss}}) \rightarrow \mathcal{S}_\bullet(\mathcal{C}_\phi^{Z\text{-ss}})$ be the relative 2-Segal groupoid associated to a stability function Z and a framing Φ on an abelian category \mathcal{C} . The universal left Hall module $\mathcal{M}(\mathcal{S}_\bullet^{\text{st-fr}}(\mathcal{C}_\phi^{Z\text{-ss}}))$ categorifies the stable framed Hall algebra representations of [38], [9] which appear in framed Donaldson-Thomas theory. \triangleleft

Example. Let $\mathcal{R}_\bullet(\mathcal{C}) \rightarrow \mathcal{S}_\bullet(\mathcal{C})$ be the relative 2-Segal groupoid associated to a proto-exact category with duality via the \mathcal{R}_\bullet -construction. For finitary exact \mathcal{C} , the universal left Hall module $\mathcal{M}(\mathcal{R}_\bullet(\mathcal{C}))$ categorifies the Hall algebra representations of [44]. Explicitly, $\mathcal{M}(\mathcal{R}_\bullet(\mathcal{C}); \mathfrak{F}_0)$ is the k -vector space with basis

$\{\mathbf{1}_{(M, \psi_M)}\}_{(M, \psi_M) \in \pi_0(\mathcal{R}_0(\mathcal{C}))}$ and left $\mathcal{H}(\mathcal{S}_\bullet(\mathcal{C}); \mathfrak{F}_0)$ -module structure

$$\mathbf{1}_U \star \mathbf{1}_{(M, \psi_M)} = \sum_{(N, \psi_N) \in \pi_0(\mathcal{R}_0(\mathcal{C}))} G_{U, M}^N \mathbf{1}_{(N, \psi_N)}.$$

The structure constant $G_{U, M}^N$ is the number of isotropic subobjects of N which are isomorphic to U and have reduction isometric to M .

Similarly, the simplicial stack version of $\mathcal{R}_\bullet(\mathcal{C})$ is relative 2-Segal and the recovers perverse sheaf theoretic [8], motivic and cohomological [45] Hall algebra representations which appear in orientifold Donaldson-Thomas theory.

Since $\mathcal{R}_\bullet(\mathcal{C})$ is 1-Segal and hence 2-Segal, we can also consider its Hall algebra $\mathcal{H}(\mathcal{R}_\bullet(\mathcal{C}); \mathfrak{F}_0)$. It has a k -basis labelled by symbols $[U \rightharpoonup (M, \psi_M)]$ with U isotropic. However, the product is not particularly interesting. Indeed,

$$[U \rightharpoonup (M, \psi_M)] \cdot [V \rightharpoonup (N, \psi_N)] = \delta_{N, M//U} [\tilde{V} \rightharpoonup (M, \psi_M)]$$

where \tilde{V} is the unique (necessarily isotropic) subobject of M with $\tilde{V}/U \simeq V$. \triangleleft

4.2. Hall monoidal module categories. We briefly describe a variant of the constructions of Section 4.1. Let k be a field of characteristic zero. Let X_\bullet be a unital 2-Segal groupoid² which is admissible with respect to weakly proper and locally proper maps. Write $\text{Fun}(X_1)$ for the category of finitely supported functors $X_1 \rightarrow \text{Vect}_k$. The span m from Theorem 4.1 induces a bifunctor

$$\otimes^X = \partial_{1*} \circ (\partial_2 \times \partial_0)^* : \text{Fun}(X_1) \otimes \text{Fun}(X_1) \rightarrow \text{Fun}(X_1).$$

It is proved in [4, Theorem 2.49] that the triple $\mathcal{H}^\otimes(X_\bullet) = (\text{Fun}(X_1), I_X, \otimes^X)$ is a monoidal category. See also [42, Proposition 5.2.6.(1)] and, more generally, [42, Theorem 5.0.1.(1)].

In the same way, we have the following relative construction. The same statement is proved in [42, Proposition 5.2.6.(2)]; a generalization is given by [42, Theorem 5.0.1.(2)].

Theorem 4.3. *Assume that $F_\bullet : Y_\bullet \rightarrow X_\bullet$ is an admissible unital relative 2-Segal groupoid. Then the left action span μ_l defines a bifunctor*

$$\otimes^Y = \partial_{1*} \circ (F_1 \times \partial_0)^* : \text{Fun}(X_1) \otimes \text{Fun}(Y_0) \rightarrow \text{Fun}(Y_0)$$

which gives $\mathcal{M}^\otimes(Y_\bullet) = (\text{Fun}(Y_0), \otimes^Y)$ the structure of a left $\mathcal{H}^\otimes(X_\bullet)$ -module category.

Example. The category $\text{Vect}_{\mathbb{F}_1}$ of finite dimensional vector spaces over \mathbb{F}_1 is proto-exact. Let $\mathcal{C}_{\mathbb{F}_1}$ be the skeleton of standard ordinals. Writing \mathfrak{S}_n for the symmetric group on n letters, we have an equivalence of groupoids

$$\mathcal{S}_1(\mathcal{C}_{\mathbb{F}_1}) \simeq \bigsqcup_{n \geq 0} B\mathfrak{S}_n.$$

Objects of $\mathcal{H}^\otimes(\mathcal{S}_\bullet(\mathcal{C}_{\mathbb{F}_1}))$ (henceforth $\mathcal{H}^\otimes(\mathcal{C}_{\mathbb{F}_1})$) are thus sequences of finite dimensional representations of symmetric groups over k , only finitely many of which are non-trivial. The monoidal product is induction of representations. Using the results of [24] we conclude that $\mathcal{H}^\otimes(\mathcal{C}_{\mathbb{F}_1})$ is equivalent to the category \mathbf{P} of polynomial functors $\text{Vect}_k \rightarrow \text{Vect}_k$.

The category $\text{Vect}_{\mathbb{F}_1}$ has a strict duality structure with $P = \text{Hom}_{\text{Vect}_{\mathbb{F}_1}}(-, \{*, 1\})$. Note that each object $V \in \text{Vect}_{\mathbb{F}_1}$ is canonically isomorphic to its dual. In particular, P preserves $\mathcal{C}_{\mathbb{F}_1}$. A symmetric form on \mathbb{F}_1^n is an element $\pi \in \mathfrak{S}_n$ which squares to the identity; conjugate such elements determine isometric symmetric forms. It follows that symmetric forms on \mathbb{F}_1^n are determined uniquely by their Witt index

²With minor modifications X_\bullet could be a simplicial object of a combinatorial model category.

$0 \leq w \leq \lfloor \frac{n}{2} \rfloor$, that is, the number of 2-cycles in any representative π . The isometry group of π , which is its centralizer in \mathfrak{S}_n , is isomorphic to $(\mathbb{Z}_2 \wr \mathfrak{S}_{\frac{n-w}{2}}) \times \mathfrak{S}_w$. We therefore obtain an equivalence of groupoids

$$\mathcal{R}_0(\mathcal{C}_{\mathbb{F}_1}) \simeq \bigsqcup_{h \geq 0} \bigsqcup_{w \geq 0} B((\mathbb{Z}_2 \wr \mathfrak{S}_h) \times \mathfrak{S}_w).$$

It follows that objects of $\mathcal{M}^\otimes(\mathcal{C}_{\mathbb{F}_1})$ are finite sequences of representations of groups of the form $(\mathbb{Z}_2 \wr \mathfrak{S}_h) \times \mathfrak{S}_w$. The left $\mathcal{H}^\otimes(\mathcal{C}_{\mathbb{F}_1})$ -action on $\mathcal{M}^\otimes(\mathcal{C}_{\mathbb{F}_1})$ is induction of representations along subgroups of the form

$$\mathfrak{S}_n \times ((\mathbb{Z}_2 \wr \mathfrak{S}_h) \times \mathfrak{S}_w) \leq (\mathbb{Z}_2 \wr \mathfrak{S}_{n+h}) \times \mathfrak{S}_w.$$

From this description we see that $\mathcal{M}^\otimes(\mathcal{C}_{\mathbb{F}_1}) = \bigoplus_{w=0}^\infty \mathcal{M}^\otimes(\mathcal{C}_{\mathbb{F}_1}; w)$ as $\mathcal{H}^\otimes(\mathcal{C}_{\mathbb{F}_1})$ -modules, the index w labelling a fixed Witt index. Moreover, we have

$$\mathcal{M}^\otimes(\mathcal{C}_{\mathbb{F}_1}; w) \simeq \mathcal{M}^\otimes(\mathcal{C}_{\mathbb{F}_1}; 0) \times \mathbf{P}^w$$

where $\mathbf{P}^w \subset \mathbf{P}$ is the full subcategory of degree w homogeneous polynomial functors. Using again [24] we find that $\mathcal{M}^\otimes(\mathcal{C}_{\mathbb{F}_1}; 0)$ is equivalent to $\mathbf{P}_{\mathbb{Z}_2}$, the category of polynomial functors $\mathbf{Vect}_k^{\mathbb{Z}_2\text{-gr}} \rightarrow \mathbf{Vect}_k$, where $\mathbf{Vect}_k^{\mathbb{Z}_2\text{-gr}}$ denotes the category of \mathbb{Z}_2 -graded finite dimensional vector spaces over k and we view $\mathbf{P}_{\mathbb{Z}_2}$ as a \mathbf{P} -module category via the forgetful functor $\mathbf{Vect}_k^{\mathbb{Z}_2\text{-gr}} \rightarrow \mathbf{Vect}_k$.

Upon passing to Grothendieck groups we obtain an isomorphism

$$K_0(\mathcal{M}^\otimes(\mathcal{C}_{\mathbb{F}_1}; w)) \simeq \left(\bigoplus_{h=0}^\infty R(\mathbb{Z}_2 \wr \mathfrak{S}_h) \right) \otimes_k R(\mathfrak{S}_w)$$

as modules over the algebra

$$K_0(\mathcal{H}^\otimes(\mathcal{C}_{\mathbb{F}_1})) \simeq \bigoplus_{n=0}^\infty R(\mathfrak{S}_n),$$

where we have written $R(-)$ for the representation ring over k . These modules have been studied in [37] and are closely related to the work of Zelevinsky [46], who studied the algebra structure on $K_0(\mathcal{M}^\otimes(\mathcal{C}_{\mathbb{F}_1}; 0))$ arising from induction of representations of wreath symmetric products. \triangleleft

Example. Let q be a prime power and consider the exact category $\mathbf{Vect}_{\mathbb{F}_q}$. Let $\mathcal{C}_{\mathbb{F}_q}$ be the skeleton consisting of \mathbb{F}_q^n , $n \geq 0$. We have an equivalence of groupoids

$$S_1(\mathcal{C}_{\mathbb{F}_q}) \simeq \bigsqcup_{n \geq 0} BGL_n(\mathbb{F}_q).$$

The category $\mathcal{H}^\otimes(\mathcal{C}_{\mathbb{F}_q})$, whose monoidal product is parabolic induction of finite dimensional representations of GL , has appeared in the work of Joyal and Street [15]. The associated algebra $K_0(\mathcal{H}^\otimes(\mathcal{C}_{\mathbb{F}_q}))$ is the ring of characters for the tower of general linear groups over \mathbb{F}_q and has been studied by Green [11] and Zelevinsky [46].

Assume now that q is odd and fix a sign $s \in \{\pm 1\}$. Take the duality $P = \text{Hom}_{\mathbf{Vect}_{\mathbb{F}_q}}(-, \mathbb{F}_q)$ with $\Theta = s \cdot \text{can}$. Identify the dual of \mathbb{F}_q^n with itself via the dual basis. Symmetric forms in $\mathbf{Vect}_{\mathbb{F}_q}$ are then orthogonal or symplectic vector spaces. It follows that we have an equivalence of groupoids

$$R_0(\mathcal{C}_{\mathbb{F}_q}) \simeq \bigsqcup_{n \geq 0} \bigsqcup_{\varepsilon \in W_n} BG_n^\varepsilon$$

where W_n is the Witt group of \mathbb{F}_q^n (trivial if $s = -1$) and $G_n^\varepsilon = O_n^\varepsilon(\mathbb{F}_q)$ if $s = +1$ and $G_n = Sp_{2n}(\mathbb{F}_q)$ if $s = -1$. The left $\mathcal{H}^\otimes(\mathcal{C}_{\mathbb{F}_q})$ -action on $\mathcal{M}^\otimes(\mathcal{C}_{\mathbb{F}_q})$ is given by parabolic induction between GL and G representations. The $K_0(\mathcal{H}^\otimes(\mathcal{C}_{\mathbb{F}_q}))$ -modules

$K_0(\mathcal{M}^\otimes(\mathcal{C}_{\mathbb{F}_q}))$ were studied by van Leeuwen [41], who showed that they are generated by cuspidal elements (with respect to the natural Hall comodule structure) of $K_0(\mathcal{M}^\otimes(\mathcal{C}_{\mathbb{F}_q}))$. \triangleleft

Heuristically, the $q \rightarrow 1$ limit of $\mathcal{M}^\otimes(\mathcal{C}_{\mathbb{F}_q})$ recovers the modules $\mathcal{M}^\otimes(\mathcal{C}_{\mathbb{F}_1}; w)$ with $w = 0, 1$, consistent with the philosophy that the $q \rightarrow 1$ limit of $\mathbf{G}(\mathbb{F}_q)$ is the Weyl group of \mathbf{G} , namely $\mathbb{Z}_2 \wr \mathfrak{S}_n$ for symplectic and even orthogonal groups and $(\mathbb{Z}_2 \wr \mathfrak{S}_n) \times \mathbb{Z}_2$ for odd orthogonal groups. The extra factor of \mathbb{Z}_2 for odd orthogonal groups reflects that the subcategories of $\mathcal{M}^\otimes(\mathcal{C}_{\mathbb{F}_q})$ with fixed Witt index are isomorphic as left $\mathcal{H}^\otimes(\mathcal{C}_{\mathbb{F}_q})$ -modules, the same statement holding also in the $q \rightarrow 1$ limit.

4.3. Modules over multivalued categories. In this section we give a categorical interpretation of relative 2-Segal simplicial sets.

Let 2-SegS be the category whose objects are unital 2-Segal simplicial sets and whose morphisms are maps of simplicial sets. Let also μCat denote the category of small multivalued categories; see [5, §3.3]. Objects of μCat are tuples $\mathfrak{X} = (\mathfrak{X}_0, \mathfrak{X}_1, m, a, e, i_{\mathfrak{X}}^l, i_{\mathfrak{X}}^r)$ consisting of sets $\mathfrak{X}_0, \mathfrak{X}_1$ with source and target maps $\partial_1, \partial_0 : \mathfrak{X}_1 \rightarrow \mathfrak{X}_0$, a composition law $m \in \text{Hom}_{\text{Span}(\text{Set})}(\mathfrak{X}_1 \times_{\mathfrak{X}_0} \mathfrak{X}_1, \mathfrak{X}_1)$, an associator isomorphism

$$a : m \circ (m \times \mathbf{1}_{\mathfrak{X}_1}) \rightarrow m \circ (\mathbf{1}_{\mathfrak{X}_1} \times m)$$

in $\text{Hom}_{\text{Span}(\text{Set})}(\mathfrak{X}_1 \times_{\mathfrak{X}_0} \mathfrak{X}_1 \times_{\mathfrak{X}_0} \mathfrak{X}_1, \mathfrak{X}_1)$ which satisfies Mac Lane coherence, a unit map $e : \mathfrak{X}_0 \rightarrow \mathfrak{X}_1$, and compatible left and right unit isomorphisms $i_{\mathfrak{X}}^l, i_{\mathfrak{X}}^r \in \text{Hom}_{\text{Span}(\text{Set})}(\mathfrak{X}_1, \mathfrak{X}_1)$.

Theorem 4.4 ([5, Theorem 3.3.6]). *The categories 2-SegS and μCat are equivalent.*

The equivalence $2\text{-SegS} \simeq \mu\text{Cat}$ is explicit; its construction will be reviewed in the proof of Theorem 4.5 below.

Consider now the relative setting. Let 2-SegRelS be the category whose objects are unital relative 2-Segal simplicial sets and whose morphisms are commutative diagrams of simplicial sets. We also require the following notion of a module over a multivalued category.

Definition. *Let \mathfrak{X} be a small multivalued category. A unital left \mathfrak{X} -module is a tuple $\mathfrak{Y} = (\mathfrak{Y}_0, \mathfrak{Y}_1, \mu, \alpha, i_{\mathfrak{Y}})$ consisting of*

- (i) *a set \mathfrak{Y}_0 together with a map $\mathfrak{Y}_0 : \mathfrak{Y}_0 \rightarrow \mathfrak{X}_0$,*
- (ii) *a left action $\mu \in \text{Hom}_{\text{Span}(\text{Set})}(\mathfrak{X}_1 \times_{\mathfrak{X}_0} \mathfrak{Y}_0, \mathfrak{Y}_0)$,*
- (iii) *a module associator isomorphism*

$$\alpha : \mu \circ (m \times \mathbf{1}_{\mathfrak{Y}_0}) \rightarrow \mu \circ (\mathbf{1}_{\mathfrak{X}_1} \times \mu)$$

in $\text{Hom}_{\text{Span}(\text{Set})}(\mathfrak{X}_1 \times_{\mathfrak{X}_0} \mathfrak{X}_1 \times_{\mathfrak{X}_0} \mathfrak{Y}_0, \mathfrak{Y}_0)$ which satisfies module theoretic Mac Lane coherence: the diagram

$$\begin{array}{ccc}
 & \mu \circ (m \circ (m \times \mathbf{1}_{\mathfrak{X}_1}) \times \mathbf{1}_{\mathfrak{Y}_0}) & \\
 \alpha \circ (\mu \times \mathbf{1}_{\mathfrak{X}_1} \times \mathbf{1}_{\mathfrak{Y}_0}) \swarrow & & \searrow \mu \circ (a \times \mathbf{1}_{\mathfrak{Y}_0}) \\
 \mu \circ (m \times \mu) & & \mu \circ ((m \circ (\mathbf{1}_{\mathfrak{X}_1} \times m)) \times \mathbf{1}_{\mathfrak{Y}_0}) \\
 \alpha \circ (\mathbf{1}_{\mathfrak{X}_1} \times \mathbf{1}_{\mathfrak{X}_1} \times \mu) \downarrow & & \downarrow \alpha \circ (\mathbf{1}_{\mathfrak{X}_1} \times m \times \mathbf{1}_{\mathfrak{Y}_0}) \\
 \mu \circ (\mathbf{1}_{\mathfrak{X}_1} \times \mu \circ (\mathbf{1}_{\mathfrak{X}_1} \times \mu)) & \xleftarrow{\mu \circ (\mathbf{1}_{\mathfrak{X}_1} \times \alpha)} & \mu \circ (\mathbf{1}_{\mathfrak{X}_1} \times (\mu \circ (m \circ \mathbf{1}_{\mathfrak{Y}_0})))
 \end{array} \tag{21}$$

commutes in $\text{Hom}_{\text{Span}(\text{Set})}(\mathfrak{X}_1 \times_{\mathfrak{X}_0} \mathfrak{X}_1 \times_{\mathfrak{X}_0} \mathfrak{X}_1 \times_{\mathfrak{X}_0} \mathfrak{Y}_0, \mathfrak{Y}_0)$, and

(iv) a unit isomorphism $i_{\mathfrak{Y}} : \mu \circ (e \times \mathbf{1}_{\mathfrak{Y}_0}) \rightarrow \mathbf{1}_{\mathfrak{Y}_0}$ in $\text{Hom}_{\text{Span}(\text{Set})}(\mathfrak{Y}_0, \mathfrak{Y}_0)$ for which the diagram

$$\begin{array}{ccc} \mu \circ (m \circ (\mathbf{1}_{\mathfrak{X}_1} \times e) \times \mathbf{1}_{\mathfrak{Y}_0}) & \xrightarrow{\alpha \circ (\mathbf{1}_{\mathfrak{X}_1} \times e \times \mathbf{1}_{\mathfrak{Y}_0})} & \mu \circ (\mathbf{1}_{\mathfrak{X}_1} \times \mu \circ (e \times \mathbf{1}_{\mathfrak{Y}_0})) \\ & \searrow \mu \circ (i_{\mathfrak{X}}^l \times \mathbf{1}_{\mathfrak{Y}_0}) & \swarrow \mu \circ (\mathbf{1}_{\mathfrak{X}_1} \times i_{\mathfrak{Y}}) \\ & \mu & \end{array} \quad (22)$$

commutes in $\text{Hom}_{\text{Span}(\text{Set})}(\mathfrak{X}_1 \times_{\mathfrak{X}_0} \mathfrak{Y}_0, \mathfrak{Y}_0)$.

Let $\phi : \mathfrak{X} \rightarrow \mathfrak{X}'$ be a morphism in μCat . Given left modules \mathfrak{Y} and \mathfrak{Y}' over \mathfrak{X} and \mathfrak{X}' , respectively, a morphism $\psi : \mathfrak{Y} \rightarrow \mathfrak{Y}'$ over ϕ consists of a map $\psi_0 : \mathfrak{Y}_0 \rightarrow \mathfrak{Y}'_0$ which satisfies $\mathfrak{F}'_0 \circ \psi_0 = \phi_0 \circ \mathfrak{F}_0$ and a morphism of spans $\psi_1 : \psi_0 \circ \mu \rightarrow \mu'(\phi_1 \times_{\phi_0} \psi_0)$. Informally, the map ψ defines a morphism of \mathfrak{X} -modules $\mathfrak{Y} \rightarrow \phi^* \mathfrak{Y}'$. The category of unital left modules over multivalued categories is denoted by $\mu\text{Cat-mod}$. We can now formulate the desired analogue of Theorem 4.4.

Theorem 4.5. *The categories 2-SegRelS and $\mu\text{Cat-mod}$ are equivalent.*

Proof. We begin by modifying the construction from the proof of [5, Theorem 3.3.6] so as to define a functor $2\text{-SegRelS} \rightarrow \mu\text{Cat-mod}$. Let $F_{\bullet} : Y_{\bullet} \rightarrow X_{\bullet}$ be a unital relative 2-Segal simplicial set. The multivalued category \mathfrak{X} associated to X_{\bullet} via Theorem 4.4 has $\mathfrak{X}_0 = X_0$ and $\mathfrak{X}_1 = X_1$ with the canonical face maps $\partial_1, \partial_0 : \mathfrak{X}_1 \rightarrow \mathfrak{X}_0$ and composition span

$$m = \{\mathfrak{X}_1 \times_{\mathfrak{X}_0} \mathfrak{X}_1 \xleftarrow{(\partial_2, \partial_0)} X_2 \xrightarrow{\partial_1} \mathfrak{X}_1\}.$$

The associator a is defined using the 2-Segal conditions associated to the triangulations of the square. Mac Lane coherence follows from the 2-Segal conditions associated decompositions of the pentagon. Let also $\mathfrak{Y}_0 = Y_0$ and $\mathfrak{F}_0 = F_0$. Define the left action span by

$$\mu = \{\mathfrak{X}_1 \times_{\mathfrak{X}_0} \mathfrak{Y}_0 \xleftarrow{(F_1, \partial_0)} Y_1 \xrightarrow{\partial_1} \mathfrak{Y}_0\}.$$

The morphisms of spans

$$\mu \circ (m \times \mathbf{1}_{\mathfrak{Y}_0}) \leftarrow \{\mathfrak{X}_1 \times_{\mathfrak{X}_0} \mathfrak{X}_1 \times_{\mathfrak{X}_0} \mathfrak{Y}_0 \leftarrow Y_2 \rightarrow \mathfrak{Y}_0\} \rightarrow \mu \circ (\mathbf{1}_{\mathfrak{X}_1} \times \mu),$$

each of which is constructed as in the proof of Theorem 4.2 and is an isomorphism by the relative 2-Segal conditions, define the module associator α . The unit isomorphism $i_{\mathfrak{Y}}$ is defined to be the inverse of the relative unit bijection $Y_{\{0\}} \rightarrow X_{\{0\}} \times_{X_{\{0,1\}}} Y_{\{0,1\}}$. Mac Lane coherence and unit compatibility are verified as in the proof of Theorem 4.2. Hence \mathfrak{Y} is a unital left \mathfrak{X} -module. That commutative squares of simplicial sets are sent to morphisms in $\mu\text{Cat-mod}$ follows immediately from the definitions.

To describe a quasi-inverse $\mu\text{Cat-mod} \rightarrow 2\text{-SegRelS}$ of the functor constructed above, let $\mathfrak{X} \in \mu\text{Cat}$ with associated unital 2-Segal simplicial set X_{\bullet} . Given a unital left \mathfrak{X} -module \mathfrak{Y} , let $Y_0 = \mathfrak{Y}_0$ and let Y_1 be the middle set of the span μ . We have canonical maps $\partial_1, \partial_0 : Y_1 \rightarrow Y_0$ and $F_i : Y_i \rightarrow X_i$, $i = 0, 1$, and, by inverting the map of sets which defines the isomorphism $i_{\mathfrak{Y}}$, a map $s_0 : Y_0 \rightarrow Y_1$. Moreover, these maps obey the 1-truncated simplicial identities. For each $n \geq 2$ define

$$\tilde{Y}_n = \varprojlim_{\sigma \in \mathcal{S}_{P_n}^{\text{Set}}} F_{\mathcal{P}}(\sigma)$$

where $\mathcal{S}_{P_n}^{\text{max}}$ is the category of symmetric embeddings of the form $\Delta^k \sqcup \Delta^k \hookrightarrow P_n$, $k \leq 2$, or $\Delta^m \hookrightarrow P_n$, $m \leq 1$ (cf. Section 2.3). Given $0 \leq i < j < k \leq n$ and

an element $\tilde{y} \in \tilde{Y}_n$, write y_{ij} and y_{ijk} for the corresponding elements of $Y_{\{i,j\}}$ and $X_{\{i,j,k\}}$, respectively. The module associator defines a collection of bijections

$$\alpha_{ijk} : X_{\{i,j,k\}} \times_{X_{\{i,k\}}} Y_{\{i,k\}} \rightarrow Y_{\{i,j\}} \times_{Y_{\{j\}}} Y_{\{j,k\}}$$

which we use to define a subset

$$Y_n = \{\tilde{y} \in \tilde{Y}_n \mid \alpha_{ijk}(y_{ijk}, y_{ik}) = (y_{ij}, y_{jk}) \text{ for all } 0 \leq i < j < k \leq n\} \subset \tilde{Y}_n.$$

Module theoretic Mac Lane coherence implies that the Y_n assemble to a 1-Segal simplicial set Y_\bullet and that the canonical maps $F_n : Y_n \rightarrow X_n$ assemble to a simplicial morphism F_\bullet . To see that F_\bullet satisfies the relative 2-Segal conditions, consider the commutative diagram

$$\begin{array}{ccc} Y_n & \xrightarrow{\quad\quad\quad} & X_n \times_{X_{\{0,n\}}} Y_{\{0,n\}} \\ \downarrow & & \downarrow \\ Y_1 \times_{Y_0} \cdots \times_{Y_0} Y_1 & \longrightarrow & X_{\{0,1,2\}} \times_{X_{\{0,2\}}} \cdots \times_{X_{\{0,n-1\}}} X_{\{0,n-1,n\}} \times_{X_{\{0,n\}}} Y_{\{0,n\}} \end{array}$$

The vertical morphisms are bijections by the 1- and 2-Segal conditions on Y_\bullet and X_\bullet , respectively. The bottom arrow is naturally interpreted as the iterated application of the module associator, relating the bracketings in which n -elements of \mathfrak{X} act on \mathfrak{Y} from right to left and from left to right, and is therefore a bijection. Hence F_\bullet is relative 2-Segal. The relative unit bijection $Y_{\{0\}} \rightarrow X_{\{0\}} \times_{X_{\{0,1\}}} Y_{\{0,1\}}$ is the inverse of $i_{\mathfrak{Y}}$. The compatibility of $i_{\mathfrak{Y}}$, $i_{\mathfrak{X}}^l$ and α implies that the higher relative unit bijections $Y_{n-1} \rightarrow X_{\{i\}} \times_{X_{\{i,i+1\}}} Y_n$ hold. Finally, it is clear that morphisms in $\mu\text{Cat-mod}$ are sent to commutative squares of simplicial sets, so the quasi-inverse is well-defined. \square

There is a semi-simplicial variant of Theorem 4.5 where $\mu\text{Cat-mod}$ is replaced with the category of left modules over multivalued semicategories, the prefix ‘semi’ indicating that unit morphisms are not omitted.

As a simple special case of the semi-simplicial variants of Theorems 4.4 and 4.5, a 2-Segal semi-simplicial set X_\bullet which satisfies $X_0 = X_1 = \text{pt}$ defines a distributive monoidal endofunctor on Set by $F \otimes F' = X_2 \times F \times F'$. The associator reduces to a bijection $a : X_2 \times X_2 \rightarrow X_2 \times X_2$ which satisfies the pentagon equation

$$a_{23} \circ a_{13} \circ a_{12} = a_{12} \circ a_{23}.$$

Conversely, a 2-Segal semi-simplicial set X_\bullet with $X_0 = X_1 = \text{pt}$ is equivalent to a bijective solution of the pentagon equation on X_2 [5, §3.7]. Explicitly, the semi-simplicial set associated to (X_2, a) has $\mathfrak{N}_0(X_2, a) = \mathfrak{N}_1(X_2, a) = \text{pt}$ while $\mathfrak{N}_n(X_2, a)$, $n \geq 2$, is the set of tuples $\{x_{ijk} \in X_2 \mid 0 \leq i < j < k \leq n\}$ which satisfy

$$a(x_{ijk}, x_{ikl}) = (x_{ijl}, x_{jkl}), \quad 0 \leq i < j < k < l \leq n.$$

We have $\mathfrak{N}_n(X_2, a) \simeq X_2^{n-1}$ for $n \geq 2$. Similarly, a relative 2-Segal semi-simplicial set $Y_\bullet \rightarrow X_\bullet$ with $Y_0 = \text{pt}$ defines a monoidal endofunctor $F \boxtimes G = Y_1 \times F \times G$ on Set which is a left \otimes -module. The module associator is a bijection $\alpha : X_2 \times Y_1 \rightarrow Y_1 \times Y_1$ which satisfies the a -pentagon equation

$$\alpha_{23} \circ \alpha_{13} \circ \alpha_{12} = \alpha_{12} \circ \alpha_{23}.$$

The corresponding relative 2-Segal semi-simplicial set has $\mathfrak{N}_0(Y_1, \alpha) = \text{pt}$ and $\mathfrak{N}_1(Y_1, \alpha) = Y_1$ while $\mathfrak{N}_n(Y_1, \alpha)$, $n \geq 2$, is the subset of

$$\mathfrak{N}_n(X_2, a) \times \{y_{ij} \in Y_1 \mid 0 \leq i < j \leq n\}$$

consisting of tuples which satisfy

$$\alpha(x_{ijk}, y_{ik}) = (y_{ij}, y_{jk}), \quad 0 \leq i < j < k \leq n.$$

Then $\mathfrak{N}_n(Y_1, \alpha) \simeq X_2^{n-1} \times Y_1$, $n \geq 1$, with structure map $\mathfrak{N}_\bullet(Y_1, \alpha) \rightarrow \mathfrak{N}_\bullet(X_2, a)$ the canonical projection.

Example. For any group G the map

$$a : G \times G \rightarrow G \times G, \quad (x, y) \mapsto (xy, y)$$

is a bijective solution to the pentagon equation on G (see [18]) and hence defines a 2-Segal semi-simplicial set $\mathfrak{N}_\bullet(G)$.

Let now $\rho : G \rightarrow \text{Aut}(M)$ be a left G -action on a set M . Then

$$\alpha : G \times M \rightarrow M \times M, \quad (x, m) \mapsto (\rho(x)m, m)$$

solves the a -pentagon equation and is a bijection if and only if ρ gives M the structure of a G -torsor. Hence, associated to each G -torsor M is a relative 2-Segal simplicial set $\mathfrak{N}_\bullet(G, M) \rightarrow \mathfrak{N}_\bullet(G)$. When $M = G$ with G acting by left multiplication the above construction recovers the left path $P^\triangleleft \mathfrak{N}_\bullet(G) \rightarrow \mathfrak{N}_\bullet(G)$. \triangleleft

Example. Let X_\bullet be a 2-Segal simplicial set. Assume that the map (∂_2, ∂_0) , as in the span (19), has finite fibres. The Hall category $H(X_\bullet)$ [5, §3.4] is then defined to be the k -linear category with objects X_0 and morphisms $\text{Hom}_{H(X_\bullet)}(a, b) = \mathfrak{F}_0(X_{a \rightarrow b})$, the vector space of compactly supported k -valued functions on

$$X_{a \rightarrow b} = \{a\} \times_{X_0} X_1 \times_{X_0} \{b\},$$

with composition defined via push-pull along the span

$$X_{b \rightarrow c} \times X_{a \rightarrow b} \leftarrow \{p \in X_2 \mid \partial_{\{0\}}(p) = a, \partial_{\{1\}}(p) = b, \partial_{\{2\}}(p) = c\} \rightarrow X_{a \rightarrow c}. \quad (23)$$

Writing $\mathbf{1}_x$ for the characteristic function of $x \in X_{a \rightarrow b}$, composition is given by $\mathbf{1}_x \cdot \mathbf{1}_{x'} = \sum_{x''} f_{x, x'}^{x''} \mathbf{1}_{x''}$, where

$$f_{x, x'}^{x''} = |\{d \in X_2 \mid \partial_2(d) = x, \partial_1(d) = x', \partial_0(d) = x''\}|.$$

If $F_\bullet : Y_\bullet \rightarrow X_\bullet$ is relative 2-Segal and the map (F_1, ∂_0) from the span (20) has finite fibres, then we can define a presheaf $F : H(X_\bullet)^{\text{op}} \rightarrow \text{Set}$ as follows. For each $a \in X_0$ let $F(a) = \mathfrak{F}_0(F_0^{-1}(a))$. Push-pull along the span

$$X_{a \rightarrow b} \times F_0^{-1}(b) \leftarrow \{q \in Y_1 \mid \partial_1(q) \in F_0^{-1}(a), \partial_0(q) \in F_0^{-1}(b)\} \rightarrow F_0^{-1}(a) \quad (24)$$

defines the action maps $\text{Hom}_{H(X_\bullet)}(a, b) \times F(b) \rightarrow F(a)$, which are associative by the relative 2-Segal conditions. Writing $\mathbf{1}_\xi \in F(a)$ for the characteristic function of $\xi \in F_0^{-1}(a)$, we have $\mathbf{1}_x \star \mathbf{1}_{\xi'} = \sum_{\xi''} g_{x, \xi'}^{\xi''} \mathbf{1}_{\xi''}$, where

$$g_{x, \xi'}^{\xi''} = |\{q \in Y_1 \mid \partial_1(q) = \xi'', \partial_0(q) = \xi', F_1(q) = x\}|.$$

\triangleleft

We briefly mention a categorification of the above example. Associated to an arbitrary 2-Segal simplicial set X_\bullet is its Hall 2-category $\mathbb{H}(X_\bullet)$ [5, §3.5.B], with objects X_0 and morphisms $\text{Hom}_{\mathbb{H}(X_\bullet)}(a, b)$ the overcategory $\text{Set}_{/X_{a \rightarrow b}}$. Composition of 1-morphisms is again defined by push-pull along the span (23). From a relative 2-Segal simplicial set $F_\bullet : Y_\bullet \rightarrow X_\bullet$ we can define a 2-functor $\mathbb{F} : \mathbb{H}(X_\bullet)^{\text{op}} \rightarrow \text{Cat}^\sqcup$ with values in the bicategory of \sqcup -semisimple categories, their additive functors and their natural transformations (see [5, §3.5.C]): for each $a \in X_0$ put $\mathbb{F}(a) = \text{Set}_{/F_0^{-1}(a)}$ and use the span (24) to define a functor

$$\mathbb{F}(a, b) : \text{Set}_{/X_{a \rightarrow b}} \rightarrow [\text{Set}_{/F_0^{-1}(b)}, \text{Set}_{/F_0^{-1}(a)}].$$

By [5, Proposition 3.5.4] the image of $\mathbb{F}(a, b)$ consists of additive functors.

Theorem 4.6.

- (1) [5, Theorem 3.5.8] *The map $X_{\bullet} \mapsto \mathbb{H}(X_{\bullet})$ defines an equivalence between the category of 2-Segal semi-simplicial sets and the category of \sqcup -semisimple semi-bicategories with equivalence classes of admissible lax 2-functors as morphisms.*
- (2) *The map $(F_{\bullet} : Y_{\bullet} \rightarrow X_{\bullet}) \mapsto (F : \mathbb{H}(X_{\bullet})^{\text{op}} \rightarrow \text{Cat}^{\sqcup})$ defines an equivalence between the category of relative 2-Segal semi-simplicial sets over X_{\bullet} and the category of Cat^{\sqcup} -valued presheaves on $\mathbb{H}(X_{\bullet})$.*

Proof. The proof is a combination of the proofs of [5, Theorem 3.5.8] and Theorem 4.5. We omit the details. \square

REFERENCES

- [1] J. Bergner, A. Osorno, V. Ozornova, M. Rovelli, and C. Scheimbauer. 2-Segal sets and the Waldhausen construction. arXiv:1609.02853, 2016.
- [2] T. Bridgeland. Stability conditions on triangulated categories. *Ann. of Math. (2)*, 166(2):317–345, 2007.
- [3] P. de Brito. Segal objects and the Grothendieck construction. arXiv:1605.00706, 2016.
- [4] T. Dyckerhoff. Higher categorical aspects of Hall algebras. arXiv:1505.06940, 2015.
- [5] T. Dyckerhoff and M. Kapranov. Higher Segal spaces I. arXiv:1212.3563, 2012.
- [6] T. Dyckerhoff and M. Kapranov. Triangulated surfaces in triangulated categories. arXiv:1306.2545, 2013.
- [7] T. Dyckerhoff and M. Kapranov. Crossed simplicial groups and structured surfaces. In *Stacks and categories in geometry, topology, and algebra*, volume 643 of *Contemp. Math.*, pages 37–110. Amer. Math. Soc., Providence, RI, 2015.
- [8] N. Enomoto. A quiver construction of symmetric crystals. *Int. Math. Res. Not.*, 12:2200–2247, 2009.
- [9] H. Franzen. On cohomology rings of non-commutative Hilbert schemes and CoHa-modules. *Math. Res. Lett.*, 23(3):804–840, 2016.
- [10] I. Gálvez-Carrillo, J. Kock, and A. Tonks. Decomposition spaces, incidence algebras and Möbius inversion I: basic theory. arXiv:1512.07573, 2015.
- [11] J. Green. The characters of the finite general linear groups. *Trans. Amer. Math. Soc.*, 80:402–447, 1955.
- [12] J. Hornbostel and M. Schlichting. Localization in Hermitian K -theory of rings. *J. London Math. Soc. (2)*, 70(1):77–124, 2004.
- [13] A. Hubery. From triangulated categories to Lie algebras: a theorem of Peng and Xiao. In *Trends in representation theory of algebras and related topics*, volume 406 of *Contemp. Math.*, pages 51–66. Amer. Math. Soc., Providence, RI, 2006.
- [14] A. Joyal. The theory of quasi-categories I. In preparation.
- [15] A. Joyal and R. Street. The category of representations of the general linear groups over a finite field. *J. Algebra*, 176(3):908–946, 1995.
- [16] A. Joyal and M. Tierney. Quasi-categories vs Segal spaces. In *Categories in algebra, geometry and mathematical physics*, volume 431 of *Contemp. Math.*, pages 277–326. Amer. Math. Soc., Providence, RI, 2007.
- [17] D. Joyce. Configurations in abelian categories. II. Ringel-Hall algebras. *Adv. Math.*, 210(2):635–706, 2007.
- [18] R. Kashaev. Quantization of Teichmüller spaces and the quantum dilogarithm. *Lett. Math. Phys.*, 43(2):105–115, 1998.
- [19] D. Kazhdan and Ya. Varshavskii. The Yoneda lemma for complete Segal spaces. *Funktsional. Anal. i Prilozhen.*, 48(2):3–38, 2014.
- [20] M. Kontsevich and Y. Soibelman. Stability structures, motivic Donaldson-Thomas invariants and cluster transformations. arXiv:0811.2435, 2008.
- [21] M. Kontsevich and Y. Soibelman. Cohomological Hall algebra, exponential Hodge structures and motivic Donaldson-Thomas invariants. *Commun. Number Theory Phys.*, 5(2):231–352, 2011.
- [22] J. Lurie. *Higher topos theory*, volume 170 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2009.
- [23] G. Lusztig. Quivers, perverse sheaves, and quantized enveloping algebras. *J. Amer. Math. Soc.*, 4(2):365–421, 1991.
- [24] I. Macdonald. Polynomial functors and wreath products. *J. Pure Appl. Algebra*, 18(2):173–204, 1980.

- [25] V. Ostrik. Module categories, weak Hopf algebras and modular invariants. *Transform. Groups*, 8(2):177–206, 2003.
- [26] R. Penner. *Decorated Teichmüller theory*. QGM Master Class Series. European Mathematical Society (EMS), Zürich, 2012.
- [27] H.-G. Quebbemann, W. Scharlau, and M. Schulte. Quadratic and Hermitian forms in additive and abelian categories. *J. Algebra*, 59(2):264–289, 1979.
- [28] D. Quillen. Higher algebraic K -theory. I. In *Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*, pages 85–147. Lecture Notes in Math., Vol. 341. Springer, Berlin, 1973.
- [29] C. Rezk. A model for the homotopy theory of homotopy theory. *Trans. Amer. Math. Soc.*, 353(3):973–1007 (electronic), 2001.
- [30] C. Ringel. Hall algebras. In *Topics in algebra, Part 1 (Warsaw, 1988)*, volume 26 of *Banach Center Publ.*, pages 433–447. PWN, Warsaw, 1990.
- [31] O. Schiffmann. Lectures on Hall algebras. In *Geometric methods in representation theory. II*, volume 24 of *Sémin. Congr.*, pages 1–141. Soc. Math. France, Paris, 2012.
- [32] M. Schlichting. Hermitian K -theory of exact categories. *J. K-Theory*, 5(1):105–165, 2010.
- [33] M. Schlichting. The Mayer-Vietoris principle for Grothendieck-Witt groups of schemes. *Invent. Math.*, 179(2):349–433, 2010.
- [34] G. Segal. Classifying spaces and spectral sequences. *Inst. Hautes Études Sci. Publ. Math.*, (34):105–112, 1968.
- [35] G. Segal. Categories and cohomology theories. *Topology*, 13:293–312, 1974.
- [36] J. Shapiro and D. Yao. Hermitian \mathcal{U} -theory of exact categories with duality functors. *J. Pure Appl. Algebra*, 109(3):323–330, 1996.
- [37] S. Shelley-Abrahamson. Hopf modules and representations of finite wreath products. *Algebr. Represent. Theory*, 2016.
- [38] Y. Soibelman. Remarks on cohomological Hall algebras and their representations. arXiv:1404.1606, 2014.
- [39] M. Szczesny. Representations of quivers over \mathbb{F}_1 and Hall algebras. *Int. Math. Res. Not. IMRN*, (10):2377–2404, 2012.
- [40] R. Thomason and T. Trobaugh. Higher algebraic K -theory of schemes and of derived categories. In *The Grothendieck Festschrift, Vol. III*, volume 88 of *Progr. Math.*, pages 247–435. Birkhäuser Boston, Boston, MA, 1990.
- [41] M. van Leeuwen. An application of Hopf-algebra techniques to representations of finite classical groups. *J. Algebra*, 140(1):210–246, 1991.
- [42] T. Walde. Hall monoidal categories and categorical modules. arXiv:1611.08241, 2016.
- [43] F. Waldhausen. Algebraic K -theory of spaces. In *Algebraic and geometric topology (New Brunswick, N.J., 1983)*, volume 1126 of *Lecture Notes in Math.*, pages 318–419. Springer, Berlin, 1985.
- [44] M. Young. The Hall module of an exact category with duality. *J. Algebra*, 446:291–322, 2016.
- [45] M. Young. Representations of cohomological Hall algebras and Donaldson-Thomas theory with classical structure groups. arXiv:1603.05401, 2016.
- [46] A. Zelevinsky. *Representations of finite classical groups: A Hopf algebra approach*, volume 869 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1981.

THE INSTITUTE OF MATHEMATICAL SCIENCES AND DEPARTMENT OF MATHEMATICS, THE CHINESE UNIVERSITY OF HONG KONG, SHATIN, HONG KONG
E-mail address: myoung@ims.cuhk.hk.edu